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Abstract

We consider the stochastic heat equation with multiplicative noise $u_t = \frac{1}{2}\Delta u + u\dot{W}$ in $\mathbb{R}_+ \times \mathbb{R}^d$, whose solution is interpreted in the mild sense. The noise $\dot{W}$ is fractional in time (with Hurst index $H \geq 1/2$), and colored in space (with spatial covariance kernel f). When $H > 1/2$, the equation generalizes the Itô-sense equation for $H = 1/2$. We prove that if f is the Riesz kernel of order α, or the Bessel kernel of order $\alpha < d$, then the sufficient condition for the existence of the solution is $d \leq 2 + \alpha$ (if $H > 1/2$), respectively $d < 2 + \alpha$ (if $H = 1/2$), whereas if f is the heat kernel or the Poisson kernel, then the equation has a solution for any d. We give a representation of the k-th order moment of the solution, in terms of an exponential moment of the "convoluted weighted" intersection local time of k independent d-dimensional Brownian motions.

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1 Introduction

The study of stochastic partial differential equations (s.p.d.e's) driven by a Gaussian noise which is white in time and has a non-trivial correlation structure in space (called "color"), constitutes now a classical line of research. These equations represent an alternative to the standard s.p.d.e.'s driven by a space-time white noise. A first step in this direction has been made in [10], where the authors identify the necessary and sufficient condition for the existence of the solution of the stochastic wave equation (in spatial dimension $d = 2$), in the space of real-valued stochastic processes. The fundamental reference in this area is Dalang's seminal article [9], in which the author gives the necessary and sufficient conditions under which various s.p.d.e 's with a white-colored noise (e.g. the wave equation, the damped heat equation, the heat equation) have a process solution, in arbitrary spatial dimension. The methods used in this article exploit the temporal martingale structure of the noise, and cannot be applied when the noise is "colored" in time. Other related references are: [36], [23], [33], [11] and [12].

Recently, there has been a growing interest in studying s.p.d.e.'s driven by a Gaussian noise which has the covariance structure of the fractional Browniam motion (fBm) in time, combined with a white (or colored) spatial covariance structure. (Recall that an *fBm* is a centered Gaussian process $(B_t)_{t\geq 0}$ with covariance $E(B_t B_s) = R_H(t,s) := (t^{2H} + s^{2H} - |t-s|^{2H})/2$, with $H \in (0,1)$. The Brownian motion is an fBm of index $H = 1/2$. We refer the reader to the expository article [27], for a comprehensive account on the fBm.) This interest comes from the large number of applications of the fBm in practice. To list only a few examples of the appearance of fractional noises in practical situations, we mention [20] for biophysics, [3] for financial time series, [13] for electrical engineering, and [7] for physics.

In the present article, we consider the stochastic heat equation with a multiplicative Gaussian noise, which is fractional (or white) in time with Hurst index $H > 1/2$ (respectively $H = 1/2$), and has a non-trivial spatial covariance structure given by a kernel f. As in [9], we assume that f is the Fourier transform of a tempered measure μ. (Note that the particular case of a spatially white noise arises when $f = \delta_0$.) More precisely, we consider the following Cauchy problem:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2}\Delta u + u\dot{W}, \quad t>0, x\in\mathbb{R}^d \\ u_{0,x} &= u_0(x), \quad x\in\mathbb{R}^d, \end{aligned} \tag{1}$$

where $u_0 \in C_b(\mathbb{R}^d)$ is non-random, and $\dot{W}$ is a formal writing for the noise $W = \{W(h); h \in \mathcal{HP}\}$ (to be introduced rigourously in Section 2).

Before discussing the multiplicative case, we recall briefly the known results related to the existence of the solution of the stochastic heat equation with additive noise:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + \dot{W}, \quad t>0, x\in\mathbb{R}^d \tag{2}$$

$$u_{0,x} \quad = \quad 0, \quad x \in \mathbb{R}^d,$$

When $H = 1/2$ and $f = \delta_0$, equation (2) admits a solution in the space of real-valued processes, if and only if $d = 1$. This phenomenon can be explained intuitively by saying that, while the Laplacian smooths, the white noise roughens (see also [16]). If the spatial dimension d is larger than 2, then the roughness effect of the white noise overcomes the smoothness influence of the Laplacian.

What happens when the space-time white noise is replaced by a noise which is fractional in time, but continues to be white in space? This situation has been studied in several papers such as [14], [24], [29], [34] and recently in [1]. In this case, a necessary and sufficient condition for the existence of the solution of (2) is $d < 4H$, which allows us to consider the cases $d = 1, 2$ or 3, for suitable values of H. This can be interpreting by saying that for $H > 1/2$, the noise roughens a little bit less, and the smoothness influence of the Laplacian overcomes the roughness of the noise. If the noise is colored in space, the conditions for the existence of the solution of (2) depend on the noise regularity in space. For example, if f the Riesz kernel of order α, or the Bessel kernel of order α, the necessary and sufficient condition for the existence of the solution of (2) is $d < 4H + \alpha$, whereas if f is the heat kernel or the Poisson kernel, the solution exists for any $d \geq 1$ and $H > 1/2$ (see [1], as well as Appendix B for a correction of the result of [1]).

Another explanation of this phenomenon is given in [16], and it is related to the local time of the stochastic processes associated with the differential operator of the s.p.d.e. In the particular case of the stochastic heat equation driven by a space-time white noise, the solution exists only in dimension $d = 1$ because this is the only case when the d-dimensional Brownian motion has a local time.

We now return to the discussion of equation (1). This equation has been studied recently in [18], when the noise is fractional in time, and white in space. In this article, it is proved that a *sufficient* condition for the existence of the solution (in the space of square-integrable processes) is $d \leq 2$: if $d = 1$, then equation (1) has a solution in any time interval $[0, T]$, but if $d = 2$, this equation has a solution only up to a critical point T_0 (i.e. it has a solution in any interval $[0, T]$, with $T < T_0$). It is not known if this condition is necessary as well. There still is a connection with the local time, in the sense that the second-order moment of the solution is equal to the exponential moment of the "weighted" intersection local time L_t of two independent d-dimensional Brownian motion B^1 and B^2, written formally as:

$$L_t := H(2H-1) \int_0^t \int_0^t |r-s|^{2H-2} \delta_0(B_r^1 - B_s^2) drds.$$

In the present article, we consider equation (1) driven by the Gaussian noise introduced in [1]. This noise is fractional in time with Hurst index $H \geq 1/2$, and colored in space, with covariance kernel f chosen among the following: the Riesz kernel, the Bessel kernel, the heat kernel, or the Poisson kernel (see Examples 2.1-2.4). The case of the fractional kernel $f(x) = \prod_{i=1}^d H_i(2H_i - 1)|x_i|^{2H_i-2}$

with $1/2 < H_i < 1$ has been examined in [17], using methods that rely on the product form of f. These methods cannot be used in the present article, since in our case (except the heat kernel), f is not of product type. For the fractional kernel, it was proved in [17] that the sufficient condition for the existence of the solution is $d < 2/(2H-1) + \sum_{i=1}^{d} H_i$.

As in the case of equation (2) with additive noise, we find that the existence of the solution depends on the roughness of the noise. If $H > 1/2$, and f is the Riesz kernel of order α, or the Bessel kernel of order $\alpha < d$ (which are "rough" kernels), then a sufficient condition for the existence of the solution is $d \leq 2+\alpha$: if $d < 2+\alpha$ the solution exists in any time interval $[0,T]$, whereas if $d = 2+\alpha$, the solution exists only up to a critical point T_0. If f is the heat or the Poisson kernel (which are "smooth" kernels), the solution exists in any time interval, for any $d \geq 1$ and $H \geq 1/2$. If f is one of the "rough" kernels mentioned above, we prove that if the solution exists, then $d < 4H+\alpha$. This shows that for $H = 1/2$, the necessary and sufficient condition for the existence of solution is $d < 2+\alpha$. It remains an open problem to identify the necessary and sufficient condition for the existence of the solution, in the case of $H > 1/2$.

The existence of the solution is connected to the "convoluted weighted" intersection local time L_t, written formally as:

$$L_t = H(2H-1) \int_0^t \int_0^t \int_{\mathbb{R}^d} |r-s|^{2H-2} \delta_0(B_r^1 - B_s^2 - y) f(y) dy dr ds.$$

More precisely, the second-order moment of the solution can be expressed as:

$$E[u_{t,x}^2] = E\left[u_0(x+B_t^1) u_0(x+B_t^2) \exp(L_t)\right].$$

As in [18], this expression can be extended to the moments of order $k \geq 2$, using k independent d-dimensional Brownian motions.

This article is organized as follows. Section 2 contains some preliminaries related to analysis on Wiener spaces. In Sections 3, we discuss the existence of the solution. In Section 4, we examine the relationship with the "convoluted weighted" intersection local time.

2 Preliminaries

We begin by describing the kernel which gives the spatial covariance of the noise. As in [9], let f be the Fourier transform of a tempered distribution μ on $\mathbb{R}^d$, i.e.

$$f(x) = \int_{\mathbb{R}^d} e^{-i\xi \cdot x} \mu(d\xi), \quad \forall x \in \mathbb{R}^d,$$

where $\xi \cdot x$ denotes the scalar product in $\mathbb{R}^d$. Let $\mathcal{P}(\mathbb{R}^d)$ be the completion of $\{1_A; A \in \mathcal{B}_b(\mathbb{R}^d)\}$, where $\mathcal{B}_b(\mathbb{R}^d)$ denotes the class of bounded Borel sets in $\mathbb{R}^d$, with respect to the inner product

$$\langle 1_A, 1_B \rangle_{\mathcal{P}(\mathbb{R}^d)} = \int_A \int_B f(x-y) dy dx.$$

We consider some examples of kernel functions f. In what follows, $|x|$ denotes the Euclidian norm of $x \in \mathbb{R}^d$.

Example 2.1 If $\mu(d\xi) = |\xi|^{-\alpha} d\xi$ for some $0 < \alpha < d$, then f is the Riesz kernel of order α:

$$f(x) = \gamma_{\alpha,d} |x|^{-d+\alpha},$$

where $\gamma_{\alpha,d} = \Gamma((d-\alpha)/2) 2^{-\alpha} \pi^{-d/2} / \Gamma(\alpha/2)$.

Example 2.2 If $\mu(d\xi) = (1+|\xi|^2)^{-\alpha/2} d\xi$ for some $\alpha > 0$, then f is the Bessel kernel of order α:

$$f(x) = \gamma'_\alpha \int_0^\infty w^{(\alpha-d)/2-1} e^{-w} e^{-|x|^2/(4w)} dw,$$

where $\gamma'_\alpha = (4\pi)^{\alpha/2} \Gamma(\alpha/2)$. In this case, $\mathcal{P}(\mathbb{R}^d)$ coincides with $\mathcal{H}^{-\alpha/2}(\mathbb{R}^d)$, the fractional Sobolev space of order $-\alpha/2$; see e.g. p.191, [15].

Example 2.3 If $\mu(d\xi) = e^{-\pi^2 \alpha |\xi|^2/2} d\xi$ for some $\alpha > 0$, then f is the heat kernel of order α:

$$f(x) = (2\pi\alpha)^{-d/2} e^{-|x|^2/(2\alpha)}.$$

Example 2.4 If $\mu(d\xi) = e^{-4\pi^2 \alpha |\xi|} d\xi$ for some $\alpha > 0$, then f is the Poisson kernel of order α:

$$f(x) = C_d \alpha (|x|^2 + \alpha^2)^{-(d+1)/2},$$

where $C_d = \pi^{-(d+1)/2} \Gamma((d+1)/2)$.

As in [1], if $H > 1/2$, we let $\mathcal{HP}$ be the Hilbert space defined as the completion of $\{1_{[0,t] \times A}; t \geq 0, A \in \mathcal{B}_b(\mathbb{R}^d)\}$ with respect to the inner product

$$\langle 1_{[0,t]\times A}, 1_{[0,s]\times B} \rangle_{\mathcal{HP}} = \alpha_H \int_0^t \int_0^s \int_A \int_B |u-v|^{2H-2} f(x-y) dy dx dv du, \quad (3)$$

where $\alpha_H = H(2H-1)$. The space $\mathcal{HP}$ is isomorphic to $\mathcal{H} \otimes \mathcal{P}(\mathbb{R}^d)$, where $\mathcal{H}$ is the completion of $\{1_{[0,t]}; t \geq 0\}$ with respect to the inner product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = \alpha_H \int_0^t \int_0^s |u-v|^{2H-2} dv du.$$

If $H = 1/2$, we let $\mathcal{HP}$ be the completion of $\{1_{[0,t] \times A}; t \geq 0, A \in \mathcal{B}_b(\mathbb{R}^d)\}$ with respect to the inner product

$$\langle 1_{[0,t]\times A}, 1_{[0,s]\times B} \rangle_{\mathcal{HP}} = (t \wedge s) \int_A \int_B f(x-y) dy dx.$$

In this case, the space $\mathcal{HP}$ is isomorphic to $L^2(\mathbb{R}_+) \otimes \mathcal{P}(\mathbb{R}^d)$.

We note that in both cases, the space $\mathcal{HP}$ may contain distributions.

Let $W = \{W(h); h \in \mathcal{HP}\}$ be a zero-mean Gaussian process, defined on a probability space $(\Omega, \mathcal{F}, P)$, with covariance

$$E(W(h)W(g)) = \langle h, g \rangle_{\mathcal{HP}}.$$

The process W introduce formally the noise perturbing the stochastic heat equation. This noise is considered to be "colored" in space, with the color given by the kernel f. If $H > 1/2$, the noise is fractional in time, whereas if $H = 1/2$ the noise is white in time.

We now introduce the basic elements of analysis on Wiener spaces, which are needed in the sequel. For a comprehensive account on this subject, we refer the reader to [26] and [28].

We begin with a brief description of the multiple Wiener (or Wiener-Itô) integral with respect to W. Let $\mathcal{F}^W$ be the σ-field generated by $\{W(h); h \in \mathcal{HP}\}$, $H_n(x)$ be the n-th order Hermite polynomial, and $\mathcal{HP}_n$ be the closed linear span of $\{H_n(W(h)); h \in \mathcal{HP}\}$ in $L^2(\Omega, \mathcal{F}^W, P)$. The space $\mathcal{HP}_n$ is called the **n-th Wiener chaos** of W.

It is known that $L^2(\Omega, \mathcal{F}^W, P) = \oplus_{n=0}^{\infty} \mathcal{HP}_n$, and hence every $F \in L^2(\Omega, \mathcal{F}^W, P)$ admits the following **Wiener chaos expansion**:

$$F = \sum_{n=0}^{\infty} J_n(F), \tag{4}$$

where $J_n : L^2(\Omega, \mathcal{F}^W, P) \to \mathcal{HP}_n$ is the orthogonal projection. By convention, $\mathcal{HP}_0 = \mathbb{R}$ and $J_0(F) = E(F)$.

For each $n \geq 1$, and for each $h \in \mathcal{HP}$ with $\|h\|_{\mathcal{HP}} = 1$, we define

$$I_n(h^{\otimes n}) = n!\ H_n(W(h)).$$

By polarization, we extend I_n to elements of the form $h_1 \otimes \ldots \otimes h_n$ (see p. 230 of [17]; e.g. $h_1 \otimes h_2 = [(h_1 + h_2)^{\otimes 2} - (h_1 - h_2)^{\otimes 2}]/4$). By linearity and continuity, we extend the definition of I_n to the space $\mathcal{HP}^{\otimes n}$. (Note that if $\{e_i; i \geq 1\}$ is a CONS in $\mathcal{HP}$, then $\{e_{i_1} \otimes \ldots \otimes e_{i_n}; i_j \geq 1\}$ is a CONS in $\mathcal{HP}^{\otimes n}$.) For any $h \in \mathcal{HP}^{\otimes n}$, we say that

$$I_n(h) := \int_{(\mathbb{R}_+ \times \mathbb{R}^d)^n} h(t_1, x_1, \ldots, t_n, x_n) dW_{t_1, x_1} \ldots dW_{t_n, x_n}$$

is the **multiple Wiener integral** of h with respect to W. We have

$$E(I_n(h)I_n(g)) = n!\ \langle \tilde{h}, \tilde{g} \rangle_{\mathcal{HP}^{\otimes n}}, \quad \forall h, g \in \mathcal{HP}^{\otimes n},$$

where $\tilde{h}(t_1, x_1, \ldots, t_n, x_n) = (n!)^{-1} \sum_{\sigma \in S_n} h(t_{\sigma(1)}, x_{\sigma(1)}, \ldots, t_{\sigma(n)}, x_{\sigma(n)})$ is the symmetrization of h with respect to the n variables $(t_1, x_1), \ldots, (t_n, x_n)$, and S_n is the set of all permutations of $\{1, \ldots, n\}$. By convention, we set $I_0(x) = x$.

The map $I_n : \mathcal{HP}^{\otimes n} \to \mathcal{HP}_n$ is surjective. Moreover, for any $F_n \in \mathcal{HP}_n$, there exists a unique $f_n \in \mathcal{HP}^{\otimes n}$ symmetric, such that $I_n(f_n) = F_n$. Using (4), we conclude that any $F \in L^2(\Omega, \mathcal{F}^W, P)$ can be written as:

$$F = \sum_{n=0}^{\infty} I_n(f_n), \tag{5}$$

where $f_0 = E(F)$ and $f_n \in \mathcal{HP}^{\otimes n}$ is symmetric and uniquely determined by F. We have:

$$E|F|^2 = \sum_{n=0}^{\infty} E|I_n(f_n)|^2 = \sum_{n=0}^{\infty} n!\ \|f_n\|^2_{\mathcal{HP}^{\otimes n}}.$$

We now introduce the stochastic integral with respect to W. Let $u = \{u_{t,x}; (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ be an $\mathcal{F}^W$-measurable square-integrable process. By (5), for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, we have

$$u_{t,x} = E(u_{t,x}) + \sum_{n=1}^{\infty} I_n(f_n(\cdot, t, x)), \tag{6}$$

where $f_n(\cdot, t, x) \in \mathcal{HP}^{\otimes n}$ is symmetric and uniquely determined by $u_{t,x}$. For each $n \geq 1$, let $\tilde{f}_n$ be the symmetrization of f_n with respect to all $n+1$ variables. Let $\tilde{f}_0 = E(u)$. We say that u is **integrable with respect to** W if $\tilde{f}_n \in \mathcal{HP}^{\otimes(n+1)}$ for every $n \geq 0$, and $\sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n)$ converges in $L^2(\Omega)$. In this case, we define the stochastic integral

$$\delta(u) := \int_0^{\infty} u_s \delta W_s = \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n)$$

Note that:

$$E|\delta(u)|^2 = \sum_{n=0}^{\infty} (n+1)!\ \|\tilde{f}_n\|^2_{\mathcal{HP}^{\otimes(n+1)}}.$$

The following alternative characterization of the operator δ is needed in the present article. Let $\mathcal{S} = \{F = f(W(h_1), \ldots, W(h_n)); f \in C_b^{\infty}(\mathbb{R}^n), h_i \in \mathcal{HP}, n \geq 1\}$ be the space of all "smooth cylindrical" random variables, where $C_b^{\infty}(\mathbb{R}^d)$ denotes the class of all bounded infinitely differentiable functions on $\mathbb{R}^n$, whose partial derivatives are also bounded. The Malliavin derivative of an element $F = f(W(h_1), \ldots, W(h_n)) \in \mathcal{S}$, with respect to W, is defined by:

$$DF := \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W(h_1), \ldots, W(h_n))h_i.$$

Note that $DF \in L^2(\Omega; \mathcal{HP})$; by abuse of notation, we write $DF = \{D_{t,x}F; (t,x) \in [0,T] \times \mathbb{R}^d\}$ even if $D_{t,x}F$ is not a function in (t,x). We endow $\mathcal{S}$ with the norm $\|F\|^2_{\mathbb{D}^{1,2}} := E|F|^2 + E\|DF\|^2_{\mathcal{HP}}$, we let $\mathbb{D}^{1,2}$ be the completion of $\mathcal{S}$ with respect to this norm. The operator D can be extended to $\mathbb{D}^{1,2}$. Then

δ : Dom $\delta \subset L^2(\Omega; \mathcal{HP}) \to L^2(\Omega)$ is the adjoint of the operator D, and is uniquely defined by the following duality relationship: $u \in$ Dom δ if and only if

$$E(F\delta(u)) = E\langle DF, u\rangle_{\mathcal{HP}}, \quad \forall F \in \mathbb{D}^{1,2}. \tag{7}$$

Note that $u \in$ Dom δ if and only if u is integrable with respect to W. (In the literature, δ is called the Skorohod integral with respect to W.)

3 Existence of the solution

In this section, we give conditions for the existence of the solution of equation (1).

Let $p_t(x)$ be the heat kernel on $\mathbb{R}^d$, i.e.

$$p_t(x) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|x|^2}{2t}\right), \quad t > 0, x \in \mathbb{R}^d.$$

For any bounded Borel function $\varphi : \mathbb{R}^d \to \mathbb{R}$, let $p_t\varphi(x) = \int_{\mathbb{R}^d} p_t(x-y)\varphi(y)dy$.

For each $t > 0$, let $\mathcal{F}_t$ be the σ-field generated by $\{W(1_{[0,s]\times A}); s \in [0,t], A \in \mathcal{B}_b(\mathbb{R}^d)\}$.

The solution of equation (1) is interpreted in the mild (or evolution) sense, using the stochastic integral introduced above. More precisely, we have the following definition.

Definition 3.1 *An $(\mathcal{F}_t)_t$-adapted square-integrable process $u = \{u_{t,x}; (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ is a* **solution to** *(1) if for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, the process*

$$\{Y^{t,x}_{s,y} = 1_{[0,t]}(s)p_{t-s}(x-y)u_{s,y};\ (s,y) \in \mathbb{R}_+ \times \mathbb{R}^d\}$$

is integrable with respect to W, and

$$u_{t,x} = p_t u_0(x) + \int_0^\infty \int_{\mathbb{R}^d} Y^{t,x}_{s,y} \delta W_{s,y}.$$

By (7), the above definition is equivalent to saying that for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, $u_{t,x} \in L^2(\Omega)$, $u_{t,x}$ is $\mathcal{F}_t$-measurable, and

$$E(u_{t,x}F) = E(F)p_t u_0(x) + E\langle Y^{t,x}, DF\rangle_{\mathcal{HP}}, \quad \forall F \in \mathbb{D}^{1,2}. \tag{8}$$

The next result establishes the existence of the solution $u = \{u_{t,x}; (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$, as a collection of random variables in $L^2(\Omega)$. As in [18] (see also [6], [21], [25], [30], [32], [35]), one can find a closed formula for the kernels $f_n(\cdot, t, x)$ appearing in the Wiener chaos expansion (6) of $u_{t,x}$.

Proposition 3.2 *In order that equation (1) possesses a solution it is necessary and sufficient that for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, we have*

$$\sum_{n=0}^\infty n!\ \|f_n(\cdot, t, x)\|^2_{\mathcal{HP}^{\otimes n}} < \infty, \tag{9}$$

where

$$f_n(t_1, x_1, \ldots, t_n, x_n, t, x) = \frac{1}{n!} \prod_{j=1}^{n} p_{t_{\rho(j+1)} - t_{\rho(j)}}(x_{\rho(j+1)} - x_{\rho(j)}) p_{t_{\rho(1)}} u_0(x_{\rho(1)}),$$

ρ denotes the permutation of $\{1, 2, \ldots, n\}$ such that $t_{\rho(1)} < t_{\rho(2)} < \ldots < t_{\rho(n)}$, $t_{\rho(n+1)} = t$ and $x_{\rho(n+1)} = x$. In this case, the solution u is unique in $L^2(\Omega)$, admits the Wiener chaos decomposition (6), and

$$E|u_{t,x}|^2 = \sum_{n=0}^{\infty} n! \, \|f_n(\cdot, t, x)\|^2_{\mathcal{HP}^{\otimes n}}. \tag{10}$$

We begin to examine condition (9). Note that

$$\alpha_n(t, x) := (n!)^2 \, \|f_n(\cdot, t, x)\|^2_{\mathcal{HP}^{\otimes n}} \le \|u_0\|^2_\infty \alpha_n(t), \tag{11}$$

with equality if $u_0 = 1$, and hence

$$E|u_{t,x}|^2 = \sum_{n=0}^{\infty} \frac{1}{n!} \alpha_n(t, x) \le \|u_0\|^2_\infty \sum_{n=0}^{\infty} \frac{1}{n!} \alpha_n(t),$$

where

$$\alpha_n(t) = \begin{cases} \alpha_H^n \int_{[0,t]^{2n}} \prod_{j=1}^n |s_j - t_j|^{2H-2} \psi^{*(n)}(\mathbf{s}, \mathbf{t}) d\mathbf{s} d\mathbf{t} & \text{if } H > 1/2 \\ \int_{[0,t]^n} \psi^{*(n)}(\mathbf{s}, \mathbf{s}) d\mathbf{s} & \text{if } H = 1/2 \end{cases} \tag{12}$$

and

$$\begin{aligned} \psi^{*(n)}(\mathbf{s}, \mathbf{t}) \quad := \quad & \int_{\mathbb{R}^{2nd}} \prod_{j=1}^{n} f(x_j - y_j) \prod_{j=1}^{n} p_{t_{\rho(j+1)} - t_{\rho(j)}}(x_{\rho(j+1)} - x_{\rho(j)}) \\ & \prod_{j=1}^{n} p_{s_{\sigma(j+1)} - s_{\sigma(j)}}(y_{\sigma(j+1)} - y_{\sigma(j)}) d\mathbf{x} d\mathbf{y}. \end{aligned} \tag{13}$$

In the above integrals, we denoted $\mathbf{s} = (s_1, \ldots, s_n)$, $\mathbf{t} = (t_1, \ldots, t_n)$, $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n)$, and we chose the permutations ρ and σ of $\{1, \ldots, n\}$ such that

$$0 < t_{\rho(1)} < t_{\rho(2)} < \ldots < t_{\rho(n)} \quad \text{and} \quad 0 < s_{\sigma(1)} < s_{\sigma(2)} < \ldots < s_{\sigma(n)}, \tag{14}$$

with $t_{\rho(n+1)} = s_{\sigma(n+1)} = t$ and $x_{\rho(n+1)} = y_{\sigma(n+1)} = x$.

Note that

$$\psi^{*(n)}(\mathbf{s}, \mathbf{t}) = \langle g_{\mathbf{s}}^{(n)}, g_{\mathbf{t}}^{(n)} \rangle_{\mathcal{P}(\mathbb{R}^d)^{\otimes n}}, \quad \forall \mathbf{t}, \mathbf{s} \in [0, t]^n,$$

where

$$\begin{aligned} g_{\mathbf{t}}^{(n)}(x_1, \ldots x_n) \quad &= \quad \prod_{j=1}^{n} p_{t_{\rho(j+1)} - t_{\rho}(j)}(x_{\rho(j+1)} - x_{\rho(j)}) \\ g_{\mathbf{s}}^{(n)}(y_1, \ldots y_n) \quad &= \quad \prod_{j=1}^{n} p_{s_{\sigma(j+1)} - s_{\sigma}(j)}(y_{\sigma(j+1)} - y_{\sigma(j)}), \end{aligned}$$

and the permutations ρ and σ are chosen such that (14) holds.

As in [1], for any $y, z \in \mathbb{R}^d$ and $u, v > 0$, we denote

$$J_f(u,v,y,z) := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p_u(x-y)p_v(x'-z)f(x-x')dxdx'.$$

Lemma 3.3 *(i) If f is the Riesz kernel of order α, or the Bessel kernel of order $\alpha < d$, then*

$$J_f(u,v,y,z) \le D_{\alpha,d}(u+v)^{-(d-\alpha)/2}, \quad \forall y, z \in \mathbb{R}^d,$$

where $D_{\alpha,d}$ is a positive constant depending on α and d.

(ii) If f is the heat kernel of order α, or the Poisson kernel of order α, then

$$J_f(u,v,y,z) \le C_{\alpha,d}, \quad \forall y, z \in \mathbb{R}^d.$$

Proof: Note that $J_f(u,v,y,z) = E[f(y-z+\sqrt{u}Y - \sqrt{v}Z)]$, where Y and Z are independent d-dimensional standard normal random vectors. We use the following inequality: (see (3.24) of [1])

$$E[e^{-|y-z+\sqrt{2u}Y-\sqrt{2v}Z|^2/(4w)}] \le \left(1+\frac{u+v}{w}\right)^{-d/2}. \tag{15}$$

(i) In the case of the Riesz kernel, this inequality has been shown in the proof of Theorem 3.13 of [1]. Suppose now that f is the Bessel kernel of order $\alpha < d$. Using (15),

$$\begin{aligned} J_f(u,v,y,z) &= \gamma'_\alpha \int_0^\infty w^{(\alpha-d)/2-1}e^{-w}E[e^{-|y-z+\sqrt{u}Y-\sqrt{v}Z|^2/(4w)}]dw \\ &\le \gamma'_\alpha \int_0^\infty w^{(\alpha-d)/2-1}e^{-w}\left(1+\frac{u+v}{2w}\right)^{-d/2}dw \\ &= \gamma'_\alpha \int_0^\infty w^{\alpha/2-1}e^{-w}\left(w+\frac{u+v}{2}\right)^{-d/2}dw = \gamma'_\alpha I_{\alpha,d}\left(\frac{u+v}{2}\right) \end{aligned}$$

where $I_{\alpha,d}(x) := \int_0^\infty w^{\alpha/2-1}e^{-w}(w+x)^{-d/2}dw$. The result follows, since

$$\begin{aligned} I_{\alpha,d}(x) &\le x^{-d/2}\int_0^x w^{\alpha/2-1}e^{-w}dw + \int_x^\infty w^{\alpha/2-1}e^{-w}(w+x)^{-d/2}dw \\ &= x^{-d/2}x^{\alpha/2}\int_0^1 y^{\alpha/2-1}e^{-xy}dy + x^{-(d-\alpha)/2}\int_1^\infty y^{\alpha/2-1}e^{-xy}(y+1)^{-d/2}dy \\ &\le x^{-(d-\alpha)/2}\int_0^1 y^{\alpha/2-1}dy + x^{-(d-\alpha)/2}\int_1^\infty y^{-(d-\alpha)/2-1}dy = K_{\alpha,d}x^{-(d-\alpha)/2}, \end{aligned}$$

where $K_{\alpha,d} = 2/\alpha + 2/(d-\alpha) = 2d/[\alpha(d-\alpha)]$, and we used the fact that $\alpha < d$.

(ii) If f is the heat kernel, using (15), we obtain:

$$\begin{aligned} J_f(u,v,y,z) &= (2\pi\alpha)^{-d/2} E[e^{-|y-z+\sqrt{u}Y-\sqrt{v}Z|^2/(2\alpha)}] \\ &\leq (2\pi\alpha)^{-d/2}\left(1+\frac{u+v}{\alpha}\right)^{-d/2} \\ &= (2\pi)^{-d/2}(\alpha+u+v)^{-d/2} \leq (2\pi\alpha)^{-d/2}. \end{aligned}$$

If f is the Poisson kernel, we have:

$$\begin{aligned} J_f(u,v,y,z) &= C_d\alpha E[(|y-z+\sqrt{u}Y-\sqrt{v}Z|^2+\alpha^2)^{-(d+1)/2}] \\ &\leq C_d\alpha(\alpha^2)^{-(d+1)/2} = C_d\alpha^{-d}. \end{aligned}$$

□

Lemma 3.4 *(i) If f is the Riesz kernel of order α, or the Bessel kernel of order $\alpha < d$, then for any $\mathbf{s},\mathbf{t} \in [0,t]^n$,*

$$\psi^{*(n)}(\mathbf{s},\mathbf{t}) \leq \left(D_{\alpha,d}2^{-(d-\alpha)/2}\right)^n [\beta(\mathbf{s})\beta(\mathbf{t})]^{-(d-\alpha)/4},$$

where $\beta(\mathbf{s}) := \prod_{j=1}^n (s_{\sigma(j+1)} - s_{\sigma(j)})$, $\beta(\mathbf{t}) := \prod_{j=1}^n (t_{\rho(j+1)} - t_{\rho(j)})$, and the permutations ρ and σ are chosen such that (14) holds.

(ii) If f is the heat kernel of order α, or the Poisson kernel of order α, then for any $\mathbf{s},\mathbf{t} \in [0,t]^n$,

$$\psi^{*(n)}(\mathbf{s},\mathbf{t}) \leq C_{\alpha,d}^n,$$

where $C_{\alpha,d}$ is a constant depending on α and d.

Proof: By the Cauchy-Schwartz inequality,

$$\psi^{*(n)}(\mathbf{s},\mathbf{t}) \leq \psi^{*(n)}(\mathbf{s},\mathbf{s})^{1/2}\psi^{*(n)}(\mathbf{t},\mathbf{t})^{1/2}.$$

To find an upper bound for $\psi^{*(n)}(\mathbf{s},\mathbf{s})$, we use Lemma 3.3 to estimate the following integrals:

$$\begin{aligned} I_j &:= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p_{u_j}(x_{\sigma(j+1)} - x_{\sigma(j)}) p_{u_j}(y_{\sigma(j+1)} - y_{\sigma(j)}) f(x_{\sigma(j)} - y_{\sigma(j)}) dx_{\sigma(j)} dy_{\sigma(j)} \\ &= J_f(u_j, u_j, x_{\sigma(j+1)}, y_{\sigma(j+1)}), \quad \text{with } u_j = s_{\sigma(j+1)} - s_{\sigma(j)},\ j = 1,\dots,n. \end{aligned}$$

(i) In this case, $I_j \leq D_{\alpha,d}[2(s_{\sigma(j+1)} - s_{\sigma(j)})]^{-(d-\alpha)/2}$ and

$$\psi^{*(n)}(\mathbf{s},\mathbf{s}) \leq D_{\alpha,d}^n 2^{-n(d-\alpha)/2}\left[\prod_{j=1}^n (s_{\sigma(j+1)} - s_{\sigma(j)})\right]^{-(d-\alpha)/2}.$$

(ii) In this case, $I_j \leq C_{\alpha,d}$ and $\psi^{*(n)}(\mathbf{s},\mathbf{s}) \leq C_{\alpha,d}^n$. □

If $H > 1/2$, it was proved in [22] that there exists $\beta_H > 0$ such that

$$\alpha_H \int_0^\infty \int_0^\infty \varphi(s)\varphi(t)|t-s|^{2H-2}dsdt \le \beta_H^2 \left(\int_0^\infty |\varphi(t)|^{1/H} dt \right)^{2H},$$

for any $\varphi \in L^{1/H}(\mathbb{R}_+)$. Hence,

$$\alpha_H^n \int_{\mathbb{R}_+^{2n}} \varphi(\mathbf{s})\varphi(\mathbf{t}) \prod_{j=1}^n |t_j - s_j|^{2H-2} d\mathbf{s}d\mathbf{t} \le \beta_H^{2n} \left(\int_{\mathbb{R}_+^n} |\varphi(\mathbf{t})|^{1/H} d\mathbf{t} \right)^{2H}, \tag{16}$$

for any $\varphi \in L^{1/H}(\mathbb{R}_+^n)$. If $H = 1/2$, we let $\beta_H = 1$.

We need the following auxiliary result.

Lemma 3.5 *Let*

$$I_n(t,h) := \int_{T_n} [(t-s_n)(s_n - s_{n-1}) \dots (s_2 - s_1)]^h d\mathbf{s},$$

where $T_n = \{\mathbf{s} = (s_1, \dots, s_n); 0 < s_1 < s_2 < \dots < s_n < t\}$. *Then* $I_n(t,h) < \infty$ *if and only if* $1 + h > 0$. *In this case,*

$$I_n(t,h) = \frac{\Gamma(1+h)^{n+1}}{\Gamma(n(1+h)+1)} t^{n(1+h)}.$$

Proof: First note that $\int_0^{s_2} (s_2 - s_1)^h ds_1 = s_2^{h+1}/(h+1)$, and then

$$\begin{aligned} \int_0^{s_3} (s_3 - s_2)^h s_2^{h+1} ds_2 &= s_3^{2h+2} \beta(h+2, h+1) \\ &= s_3^{2(h+1)} \beta((h+1)+1, h+1)) \end{aligned}$$

where $\beta(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$ is the beta function, and we used the change of variables $s_2/s_3 = z$. In this way, the integral $I_n(t,h)$ becomes

$$\begin{aligned} I_n(t,h) &= \frac{1}{h+1} \beta((h+1)+1, h+1) \dots \beta((n-2)(h+1)+1, h+1) \\ &\quad \int_0^t s_n^{(n-1)(h+1)} (t-s_n)^h ds_n \\ &= t^{n(h+1)} \beta((h+1)+1, h+1) \dots \beta((n-1)(h+1)+1, h+1). \end{aligned}$$

Using the fact that $\beta(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ for $a, b > 0$ and $\Gamma(z+1) = z\Gamma(z)$ for any $z > 0$, we obtain the desired conclusion. $\square$

Using Lemma 3.4, Lemma 3.5 and (16), we obtain the following estimate for $\alpha_n(t)$.

Proposition 3.6 *Suppose that* $H \ge 1/2$ *and let* $\alpha_n(t)$ *be given by (12).*

(i) If f is the Riesz kernel of order α, or the Bessel kernel of order $\alpha < d$, and

$$H > \frac{d-\alpha}{4}\ , \tag{17}$$

then

$$\alpha_n(t) \leq C^*_{H,d,\alpha} D(t)^n\ (n!)^{(d-\alpha)/2}, \quad \textit{for any } t>0, n \geq 1,$$

where $C^*_{H,d,\alpha} > 0$ is a constant depending on H, d, α, and

$$D(t) = D_{\alpha,d} 2^{-(d-\alpha)/2} \beta_H^2\, \Gamma\left(1-\frac{d-\alpha}{4H}\right)^{2H} \left(1-\frac{d-\alpha}{4H}\right)^{-[2H-(d-\alpha)/2]} t^{2H-(d-\alpha)/2}.$$

(ii) If f is the heat kernel of order α, or the Poisson kernel of order α, then

$$\alpha_n(t) \leq C(t)^n, \quad \textit{for any } t>0 \textit{ and } n \geq 1,$$

where $C(t) = C_{\alpha,d} t^{2H}$.

Remark 3.7 Proposition 3.6 shows that $\sum_n \alpha_n(t)/n!$ grows exponentially in time in some cases, and faster than exponentially in other cases.

Proof of Proposition 3.6: We only give the proof in the case $H > 1/2$, the case $H = 1/2$ being similar. We use the definition (12) of $\alpha_n(t)$.

(i) Let $h = -(d-\alpha)/(4H)$. By Lemma 3.4.(i) and (16), we obtain:

$$\begin{aligned}
\alpha_n(t) &\leq \left(D_{\alpha,d} 2^{-(d-\alpha)/2}\right)^n \alpha_H^n \int_{([0,t]^2)^n} \prod_{j=1}^n |t_j - s_j|^{2H-2} [\beta(\mathbf{s})\beta(\mathbf{t})]^{-(d-\alpha)/4} d\mathbf{s} d\mathbf{t} \\
&\leq \left(D_{\alpha,d} 2^{-(d-\alpha_f)/2}\right)^n \beta_H^{2n} \left(\int_{[0,t]^n} \beta(\mathbf{s})^{-(d-\alpha)/(4H)} d\mathbf{s}\right)^{2H} \\
&= \left(D_{\alpha,d} 2^{-(d-\alpha)/2} \beta_H^2\right)^n (n!)^{2H} I_n(t,h)^{2H}.
\end{aligned}$$

Using Lemma 3.5, we obtain:

$$\begin{aligned}
\alpha_n(t) &\leq \Gamma(1+h)^{2H} \left(D_{\alpha,d} 2^{-(d-\alpha)/2}\ \beta_H^2\right)^n\ (n!)^{2H} \left\{\frac{\Gamma(1+h)^n}{\Gamma(n(1+h)+1)} t^{n(1+h)}\right\}^{2H} \\
&= \Gamma(1+h)^{2H} \left\{D_{\alpha,d} 2^{-(d-\alpha)/2}\ \beta_H^2 \Gamma(1+h)^{2H}\, t^{2H(1+h)}\right\}^n \left(\frac{n!}{\Gamma(n(1+h)+1)}\right)^{2H}.
\end{aligned}$$

The result follows by using (3.19) of [18].

(ii) By Lemma 3.4.(ii),

$$\alpha_n(t) \leq C^n_{\alpha,d} \alpha_H^n \int_{[0,t]^{2n}} \prod_{j=1}^n |s_j - t_j|^{2H-2} d\mathbf{s} d\mathbf{t} = C^n_{\alpha,d}\ t^{2Hn} = C(t)^n.$$

□

Using Proposition 3.6, we examine the existence of the solution of equation (1). The next result is an extension of Proposition 4.3 of [18] to the case of a colored noise W.

Proposition 3.8 *(i) Let f be the Riesz kernel of order α, or the Bessel kernel of order $\alpha < d$. Suppose that either*

$$H > 1/2 \quad \textit{and} \quad d \leq 2 + \alpha, \tag{18}$$

or

$$H = 1/2 \quad \textit{and} \quad d < 2 + \alpha. \tag{19}$$

Then (1) has a unique solution in $[0, T] \times \mathbb{R}^d$, provided that $T < T_0$ where

$$T_0 = \begin{cases} \left\{ \left(1 - \frac{1}{2H}\right) D_{\alpha,d} 2^{-1} \beta_H^2 \Gamma\left(1 - \frac{1}{2H}\right)^{2H} \right\}^{-1/(2H-1)} & \textit{if } d = 2 + \alpha \\ \infty & \textit{if } d < 2 + \alpha \end{cases} \tag{20}$$

(ii) Let $H \geq 1/2$, and f be the heat kernel of order α, or the Poisson kernel of order α. Then (1) has a unique solution in $\mathbb{R}_+ \times \mathbb{R}^d$.

Remark 3.9 Either one of conditions (18) or (19) is stronger that (17).

Remark 3.10 Proposition 3.8 shows that in the case $H = 1/2$, the dimension $d = 2 + \alpha$ cannot be attained.

Proof of Proposition 3.8: We apply Proposition 3.2, using Proposition 3.6.
(i) We have:

$$\sum_{n=0}^{\infty} n! \, \|f_n(\cdot, t, x)\|^2_{\mathcal{HP}^{\otimes n}} \leq \|u_0\|^2_\infty \sum_{n=0}^{\infty} \frac{1}{n!} \alpha_n(t) \leq \|u_0\|^2_\infty C^*_{H,\alpha,d} \sum_{n=0}^{\infty} \frac{D(t)^n}{(n!)^{1-(d-\alpha)/2}}.$$

If $d - \alpha = 2$, then the last sum is finite if $D(t) < 1$, which is equivalent to saying that $t < T_0$. If $d - \alpha < 2$, then the last sum is finite for any $t > 0$, by Stirling's formula and D'Alembert criterion.
(ii) We have:

$$\sum_{n=0}^{\infty} n! \, \|f_n(\cdot, t, x)\|^2_{\mathcal{HP}^{\otimes n}} \leq \|u_0\|^2_\infty \sum_{n=0}^{\infty} \frac{1}{n!} \alpha_n(t) \leq \|u_0\|^2_\infty \sum_{n=0}^{\infty} \frac{C(t)^n}{n!} < \infty.$$

□

The next result shows that (17) is a necessary condition for the existence of the solution.

Proposition 3.11 *Suppose that $H \geq 1/2$ and f is either the Riesz kernel or order α, or the Bessel kernel of order α. If equation (1) with $u_0 = 1$ has a solution in $\mathbb{R}_+ \times \mathbb{R}^d$, then (17) holds.*

Proof: Note that $E\,|u_{t,x}|^2 = \sum_{n=0}^{\infty} \alpha_n(t)/n! < \infty$ implies that $\alpha_1(t) < \infty$, which in turn implies (17) (see Appendix A). □

Remark 3.12 Proposition 3.8 and Proposition 3.11 show that, if $H = 1/2$ and f is the Riesz kernel of order α, or the Bessel kernel of order $\alpha < d$, then the condition $d < \alpha + 2$ is necessary and sufficient for the existence of the solution of (1). It remains an open problem to see if this condition is necessary, when $H > 1/2$. To resolve this issue, one needs to develop a full analysis of the range of $\alpha_n(t)$, which would include the identification of suitable lower bounds. Such analysis will be the subject of future investigations.

Remark 3.13 The case $H < \frac{1}{2}$ also constitutes an interesting line of investigation, which will be pursued in a subsequent article. We mention that in this case even the stochastic heat equation with linear additive fractional-colored noise has not been solved. The technical difficulties that appear here are related to the structure of the space $\mathcal{HP}$ and the lack of the expression of the scalar product in this space as (3). Indeed, when $H < \frac{1}{2}$, assuming that the noise $W(t,x)$ is defined for $t \in [0,T]$ and $x \in \mathbb{R}^d$, the space $\mathcal{HP}$ can be described as the space of measurable functions $\varphi(s,x)$, $s \in [0,T], x \in \mathbb{R}^d$ such that $K^*\varphi \in L^2([0,T]) \otimes \mathcal{P}(\mathbb{R}^d)$, where

$$K^*\varphi(s,x) = K(T,s)\varphi(s,x) + \int_0^T \int_s^T ((\varphi(r,x) - \varphi(s,x))\, \partial_1 K(r,s) dr.$$

So, it is necessary to use the transfer operator K^* and to check (in the case of the additive noise) that $K^* g_{t,x} \in L^2([0,T]) \otimes \mathcal{P}(\mathbb{R}^d)$ where $g_{t,x}(s,y) = p_{t-s}(x - y)1_[0,t]$ which is in principle rather technical (in the of the stochastic heat equation with multiplicative fractional-colored noise, one needs to deal with the tensor product operator $(K^*)^{\otimes n}$ which has a complicated expression).

Remark 3.14 If $H = 1/2$ and f is an arbitrary kernel, it was proved in [9] (using different methods) that the sufficient condition for the existence of the solution in $\mathbb{R}_+ \times \mathbb{R}^d$ of (1) with vanishing initial conditions (i.e. $u_0 = 0$), is:

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1 + |\xi|^2} < \infty. \tag{21}$$

(see Remark 14 of [9]). When f is the Riesz kernel of order α, or the Bessel kernel of order α, (21) holds if and only if $d < \alpha + 2$. Combining this with Remark 3.12, we conclude that, in the case of the two kernels, condition (21) is also necessary for the existence of the solution. For an arbitrary kernel f, it is not known if (21) remains a necessary condition for the existence of the solution.

If $H > 1/2$ and f is the Riesz kernel of order α, or the Bessel kernel of order α, the necessary and sufficient for the existence of the solution of the stochastic heat equation with linear additive noise is $d < 4H + \alpha$, whereas if f is heat or the Poisson kernel, this equation has a solution for any d (see [1] and [2]).

4 Relationship with the Local Time

In this section, we identify a random variable L_t, defined formally as a "convoluted intersection local time" of two independent d-dimensional standard Brownian motions, such that

$$\alpha_n(t) = E(L_t^n), \quad \forall n \geq 1. \tag{22}$$

An immediate consequence of (10), (11) and (22) is that the second moment of $u_{t,x}$ is bounded by the exponential moment of L_t:

$$E|u_{t,x}|^2 \leq \|u_0\|_\infty^2 \sum_{n=0}^{\infty} \frac{1}{n!}\alpha_n(t) = \|u_0\|_\infty^2 \sum_{n=0}^{\infty} \frac{1}{n!}E(L_t^n) = \|u_0\|_\infty^2 E(e^{L_t}),$$

with equality if $u_0 = 1$.

To show (22), we approximate $\alpha_n(t)$ by $\{\alpha_{n,\varepsilon}(t)\}_{\varepsilon>0}$, when $\varepsilon \to 0$, where the constants $\alpha_{n,\varepsilon}(t)$ are chosen such that:

$$\alpha_{n,\varepsilon}(t) = E(L_{t,\varepsilon}^n), \quad \forall n \geq 1,$$

for a certain random variable $L_{t,\varepsilon}$.

To identify the approximation constants $\alpha_{n,\varepsilon}(t)$, we recall the definition (12), which says that $\alpha_n(t)$ is the weighted integral of the function $\psi^{*(n)}(\mathbf{s},\mathbf{t})$. The next lemma gives the exact calculation for the integrand $\psi^{*(n)}(\mathbf{s},\mathbf{t})$.

Lemma 4.1 *We have:*

$$\psi^{*(n)}(\mathbf{s},\mathbf{t}) = (2\pi)^{-nd} \int_{(\mathbb{R}^d)^n} \exp\left\{-\frac{1}{2}\sum_{j,k=1}^{n} \sigma_{jk}^* \xi_j \cdot \xi_k\right\} \mu(d\xi_1)\ldots\mu(d\xi_n),$$

where $\sigma_{jk}^* := (t-s_j) \wedge (t-s_k) + (t-t_j) \wedge (t-t_k)$.

Remark 4.2 Lemma 4.1 gives a generalization -and a minor correction- to Lemma 4.2 of [18]. The correction refers to the fact that the result of [18] is stated incorrectly with the constant $\sigma_{jk} = s_j \wedge s_k + t_j \wedge t_k$, instead of σ_{jk}^*. However, a trivial change of variables $s_j' := t - s_j, t_j' := t - t_j$ in the definition (12) of $\alpha_n(t)$ shows that this minor error does not affect the calculation of $\alpha_n(t)$. We have indeed:

$$\alpha_n(t) = \begin{cases} \alpha_H^n \int_{[0,t]^{2n}} \prod_{j=1}^n |s_j - t_j|^{2H-2} \psi^{(n)}(\mathbf{s},\mathbf{t}) d\mathbf{s} d\mathbf{t} & \text{if } H > 1/2 \\ \int_{[0,t]^n} \psi^{(n)}(\mathbf{s},\mathbf{s}) d\mathbf{s} & \text{if } H = 1/2 \end{cases} \tag{23}$$

with

$$\begin{aligned} \psi^{(n)}(\mathbf{s},\mathbf{t}) &= (2\pi)^{-nd} \int_{(\mathbb{R}^d)^n} \exp\left\{-\frac{1}{2}\sum_{j,k=1}^{n} \sigma_{jk} \xi_j \cdot \xi_k\right\} \mu(d\xi_1)\ldots\mu(d\xi_n) \\ &= \psi^{*(n)}(t\mathbf{1} - \mathbf{s}, t\mathbf{1} - \mathbf{t}), \quad \text{where } \mathbf{1} = (1,\ldots,1) \in \mathbb{R}^n. \end{aligned}$$

Proof of Lemma 4.1: Note that

$$\langle \varphi, \psi \rangle_{\mathcal{P}(\mathbb{R}^d)^{\otimes n}} = (2\pi)^{-nd} \int_{(\mathbb{R}^d)^n} \mathcal{F}\varphi(\xi_1, \ldots, \xi_n) \overline{\mathcal{F}\psi(\xi_1, \ldots, \xi_n)} \mu(d\xi_1) \ldots \mu(d\xi_n),$$

where $\mathcal{F}$ denotes the Fourier transform. Hence,

$$\begin{aligned} \psi^{*(n)}(\mathbf{s}, \mathbf{t}) &= \langle g_{\mathbf{s}}^{(n)}, g_{\mathbf{t}}^{(n)} \rangle_{\mathcal{P}(\mathbb{R}^d)^{\otimes n}} \\ &= (2\pi)^{-nd} \int_{(\mathbb{R}^d)^n} \mathcal{F} g_{\mathbf{s}}^{(n)}(\xi_1, \ldots, \xi_n) \overline{\mathcal{F} g_{\mathbf{t}}^{(n)}(\xi_1, \ldots, \xi_n)} \mu(d\xi_1) \ldots \mu(d\xi_n). \end{aligned}$$

It was shown in the proof of Lemma 4.2 of [18] that:

$$\begin{aligned} \mathcal{F} g_{\mathbf{s}}(\xi_1, \ldots, \xi_n) &= E\left[\prod_{j=1}^n e^{-i\xi_j \cdot [x - (B_t^1 - B_{s_j}^1)]} \right] \\ \mathcal{F} g_{\mathbf{t}}(\xi_1, \ldots, \xi_n) &= E\left[\prod_{j=1}^n e^{-i\xi_j \cdot [x - (B_t^2 - B_{t_j}^2)]} \right], \end{aligned}$$

where $B^1 = (B_t^1)_{t \geq 0}$ and $B^2 = (B_t^2)_{t \geq 0}$ are independent d-dimensional standard Brownian motions. Hence,

$$\psi^{*(n)}(\mathbf{s}, \mathbf{t}) = (2\pi)^{-nd} \int_{(\mathbb{R}^d)^n} E\left[\prod_{j=1}^n e^{-i\xi_j \cdot [(B_{s_j}^1 - B_t^1) - (B_{t_j}^2 - B_t^2)]} \right] \mu(d\xi_1) \ldots \mu(d\xi_n)$$

We begin to evaluate the integrand of the above integral. We denote $\xi_j = (\xi_{j,1}, \ldots, \xi_{j,d})$, $B_t^1 = (B_{t,1}^1, \ldots, B_{t,d}^1)$ and $B_t^2 = (B_{t,1}^2, \ldots, B_{t,d}^2)$. We observe that for any $j = 1, \ldots, n$ fixed, the random variables

$$(B_{s_j,l}^1 - B_{t,l}^1) - (B_{t_j,l}^2 - B_{t,l}^2), \quad l = 1, \ldots, d, \quad \text{are i.i.d.},$$

with the same distribution as $(b_{s_j}^1 - b_t^1) - (b_{t_j}^2 - b_t^2)$, where $b^1 = (b_t^1)_{t \geq 0}$ and $b^2 = (b_t^2)_{t \geq 0}$ are independent 1-dimensional standard Brownian motions. Hence,

$$\begin{aligned} E\left[\prod_{j=1}^n e^{-i\xi_j \cdot [(B_{s_j}^1 - B_t^1) - (B_{t_j}^2 - B_t^2)]} \right] &= \prod_{l=1}^d E\left[\prod_{j=1}^n e^{-i\xi_{j,l} [(B_{s_j,l}^1 - B_{t,l}^1) - (B_{t_j,l}^2 - B_{t,l}^2)]} \right] \\ = \prod_{l=1}^d E\left[\prod_{j=1}^n e^{-i\xi_{j,l} [(b_{s_j}^1 - b_t^1) - (b_{t_j}^2 - b_t^2)]} \right] &= \prod_{l=1}^d \exp\left\{ -\frac{1}{2} \sum_{j,k=1}^n \sigma_{jk}^* \xi_{j,l} \xi_{k,l} \right\} \\ = \exp\left\{ -\frac{1}{2} \sum_{j,k=1}^n \sigma_{jk}^* \xi_j \cdot \xi_k \right\}, \end{aligned}$$

where for the second last equality we used the fact that the vector

$$((b^1_{s_1} - b^1_t) - (b^2_{t_1} - b^2_t), \dots, (b^1_{s_n} - b^1_t) - (b^2_{t_n} - b^2_t))$$

has a normal distribution with mean zero and covariance matrix $(\sigma^*_{jk})_{1 \le j,k \le n}$. This concludes the proof of the lemma. $\Box$

In what follows, we use the alternative definition (23) of $\alpha_n(t)$, given in Remark 4.2. The idea is to find a suitable approximation for the integrand $\psi^{(n)}(\mathbf{s}, \mathbf{t})$, by replacing the Dirac function $\delta_0(x)$ with the heat kernel $p_\varepsilon(x)$. This approximation turns out to be:

$$\psi^{(n)}_\varepsilon(\mathbf{s}, \mathbf{t}) := E\left[\int_{(\mathbb{R}^d)^n} p_\varepsilon(B^1_{s_1} - B^2_{t_1} - y_1) \dots p_\varepsilon(B^1_{s_n} - B^2_{t_n} - y_n) f(y_1) \dots f(y_n) d\mathbf{y}\right],$$

where $B^1 = (B^1_t)_{t \ge 0}$ and $B^2 = (B^2_t)_{t \ge 0}$ are independent d-dimensional standard Brownian motions, and we denote $\mathbf{y} = (y_1, \dots, y_n)$.

More precisely, we have the following result.

Lemma 4.3 *Suppose that $\mu(d\xi) = g(\xi)d\xi$, i.e. $f = \mathcal{F}g$. Then*

$$\psi^{(n)}_\varepsilon(\mathbf{s}, \mathbf{t}) = (2\pi)^{-nd} \int_{(\mathbb{R}^d)^n} \exp\left\{-\frac{1}{2}\sum_{j,k=1}^n \sigma_{jk}\xi_j \cdot \xi_k - \frac{\varepsilon}{2}\sum_{j=1}^n |\xi_j|^2\right\} \mu(d\xi_1) \dots \mu(d\xi_n).$$

Proof: We first calculate the inverse Fourier transform of $p_\varepsilon * f$:

$$\mathcal{F}^{-1}(p_\varepsilon * f)(\xi) = \mathcal{F}^{-1}p_\varepsilon(\xi)\ \mathcal{F}^{-1}f(\xi) = (2\pi)^{-d} e^{-\varepsilon|\xi|^2/2} g(\xi).$$

This shows that $(p_\varepsilon * f)(x) = (2\pi)^{-d}\mathcal{F}[e^{-\varepsilon|\xi|^2/2}g(\xi)](x)$, i.e.

$$\int_{\mathbb{R}^d} \frac{1}{(2\pi\varepsilon)^{d/2}} e^{-|x-y|^2/(2\varepsilon)} f(y)dy = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{-i\xi\cdot x} e^{-\varepsilon|\xi|^2/2} g(\xi) d\xi. \qquad (24)$$

Using (24) with $x = B^1_{s_j} - B^2_{t_j}$, we obtain:

$$\int_{\mathbb{R}^d} \frac{1}{(2\pi\varepsilon)^{d/2}} e^{-|B^1_{s_j} - B^2_{t_j} - y_j|^2/(2\varepsilon)} f(y_j)dy_j = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{-i\xi_j\cdot(B^1_{s_j} - B^2_{t_j})} e^{-\varepsilon|\xi_j|^2/2} g(\xi_j) d\xi_j.$$

Therefore,

$$\begin{aligned}
\psi^{(n)}_\varepsilon(\mathbf{s}, \mathbf{t}) &= E\left[\prod_{j=1}^n \int_{\mathbb{R}^d} \frac{1}{(2\pi\varepsilon)^{d/2}} e^{-|B^1_{s_j} - B^2_{t_j} - y_j|^2/(2\varepsilon)} f(y_j)dy_j\right] \\
&= (2\pi)^{-nd} E\left[\prod_{j=1}^n \int_{\mathbb{R}^d} e^{-i\xi_j\cdot(B^1_{s_j} - B^2_{t_j})} e^{-\varepsilon|\xi_j|^2/2} g(\xi_j) d\xi_j\right] \\
&= (2\pi)^{-nd} \int_{(\mathbb{R}^d)^n} E\left[\prod_{j=1}^n e^{-i\xi_j\cdot(B^1_{s_j} - B^2_{t_j})}\right] e^{-\varepsilon\sum_{j=1}^n |\xi_j|^2/2} \mu(d\xi_1)\dots\mu(d\xi_n) \\
&= (2\pi)^{-nd} \int_{(\mathbb{R}^d)^n} e^{-\sum_{j,k=1}^n \sigma_{jk}\xi_j\cdot\xi_k/2} e^{-\varepsilon\sum_{j=1}^n |\xi_j|^2/2} \mu(d\xi_1)\dots\mu(d\xi_n).
\end{aligned}$$

$\square$

Remark 4.4 Note that the function $h = p_\varepsilon * f$ is continuous. To see this, let $(x_n)_n \subset \mathbb{R}^d$ such that $x_n \to x$. By (24) and the dominated convergence theorem, it follows that $h(x_n) \to h(x)$. To justify this, note that $|e^{-i\xi\cdot x_n}e^{-\varepsilon|\xi|^2/2}g(\xi)| \leq e^{-\varepsilon|\xi|^2/2}g(\xi)$ for any $\xi \in \mathbb{R}^d, n \geq 1$ and

$$(2\pi)^{-2d}\int_{\mathbb{R}^d} e^{-\varepsilon|\xi|^2/2}g(\xi)d\xi = \int_{\mathbb{R}^d} |\mathcal{F}p_{\varepsilon/2}(\xi)|^2 g(\xi)d\xi = \|p_{\varepsilon/2}\|^2_{\mathcal{P}(\mathbb{R}^d)} < \infty,$$

since $p_{\varepsilon/2} \in L^2(\mathbb{R}^d) \subset \mathcal{P}(\mathbb{R}^d)$.

We are now ready to define the approximation constants $\alpha_{n,\varepsilon}(t)$:

$$\alpha_{n,\varepsilon}(t) = \begin{cases} \alpha_H^n \int_{[0,t]^{2n}} \prod_{j=1}^n |t_j - s_j|^{2H-2}\psi_\varepsilon^{(n)}(\mathbf{s},\mathbf{t})d\mathbf{s}d\mathbf{t} & \text{if } H > 1/2 \\ \int_{[0,t]^n} \psi_\varepsilon^{(n)}(\mathbf{s},\mathbf{s})d\mathbf{s} & \text{if } H = 1/2 \end{cases}.$$

Note that $\alpha_{n,\varepsilon}(t) = E(L^n_{t,\varepsilon})$, where

$$L_{t,\varepsilon} := \begin{cases} \alpha_H \int_0^t\int_0^t\int_{\mathbb{R}^d} |r-s|^{2H-2}p_\varepsilon(B_r^1 - B_s^2 - y)f(y)dydrds & \text{if } H > 1/2 \\ \int_0^t\int_{\mathbb{R}^d} p_\varepsilon(B_s^1 - B_s^2 - y)f(y)dyds & \text{if } H = 1/2 \end{cases}$$

The random variable $L_{t,\varepsilon}$ is an approximation of the "convoluted intersection local time" L_t, written formally as:

$$L_t = \begin{cases} \alpha_H \int_0^t\int_0^t\int_{\mathbb{R}^d} |r-s|^{2H-2}\delta_0(B_r^1 - B_s^2 - y)f(y)dydrds & \text{if } H > 1/2 \\ \int_0^t\int_{\mathbb{R}^d} \delta_0(B_s^1 - B_s^2 - y)f(y)dyds & \text{if } H = 1/2 \end{cases}$$

Remark 4.5 We mention that this approximation procedure has been intensively used in several papers dealing with the chaos expansion of the local time and Tanaka's formulas for Brownian motion (see e.g. [31]) or fractional Brownian motion (see [8]). Recall that the local time of the Brownian motion can be formally written as $L(t,x) = \int_0^t \delta_0(B_s - x)ds$ where δ_0 is the delta Dirac function. Usually, to obtain the chaos expansion of $L(t,x)$ one approximates $\delta_0(B_s - x)$ by the Gaussian kernel p_ϵ.

More generally (and for the sake of a result encountered later in the sequel), if $\eta : [0,t]^2 \to \mathbb{R}_+$ is an arbitrary function such that $\eta(r,s) = \eta(t-r,t-s)$ for all $r,s \in [0,t]$, we define

$$L_{t,\varepsilon}(\eta) := \int_0^t\int_0^t\int_{\mathbb{R}^d} \eta(r,s)p_\varepsilon(B_r^1 - B_s^2 - y)f(y)dydrds.$$

Then $\alpha_{n,\varepsilon}(t,\eta) = E(L_{t,\varepsilon}(\eta)^n)$, where $\alpha_{n,\varepsilon}(t,\eta) := \int_{[0,t]^{2n}} \prod_{j=1}^n \eta(s_j,t_j)\psi_\varepsilon^{(n)}(\mathbf{s},\mathbf{t})d\mathbf{s}d\mathbf{t}$. Let

$$\alpha_n(t,\eta) := \int_{[0,t]^{2n}} \prod_{j=1}^n \eta(s_j,t_j)\psi^{(n)}(\mathbf{s},\mathbf{t})d\mathbf{s}d\mathbf{t}, \tag{25}$$

and note that $\alpha_{n,\varepsilon}(t,\eta) \le \alpha_n(t,\eta)$ for all $\varepsilon > 0$. Note that $\alpha_{n,\varepsilon}(t,\eta) \le \alpha_{n,\varepsilon'}(t,\eta)$ if $0 < \varepsilon' < \varepsilon$.

Lemma 4.6 *Let $t > 0$ be arbitrary. a) If $\alpha_2(t,\eta) < \infty$, then*

$$\lim_{\varepsilon,\delta\downarrow 0} E(L_{t,\varepsilon}(\eta)L_{t,\delta}(\eta)) = \alpha_2(t,\eta), \tag{26}$$

and there exists a random variable $L_t(\eta) := \lim_{\varepsilon\downarrow 0} L_{t,\varepsilon}(\eta)$ in $L^2(\Omega)$.

b) If $\alpha_n(t,\eta) < \infty$ for all $n \ge 1$, then the random variable L_t, defined in part a), is p-integrable for any $p \ge 2$, and

$$\lim_{\varepsilon\downarrow 0} E|L_{t,\varepsilon}(\eta) - L_t(\eta)|^p = 0, \quad \textit{for all } p \ge 2. \tag{27}$$

In particular, $E(L_t(\eta)^n) = \lim_{\varepsilon\downarrow 0} E(L_{t,\varepsilon}(\eta)^n) = \alpha_n(t,\eta)$ for all $n \ge 1$.

The random variable $L_t(\eta)$ defined in Lemma 4.6 depends on B^1 and B^2, and could be denoted by:

$$L_t^{B^1,B^2}(\eta) = \int_0^t \int_0^t \int_{\mathbb{R}^d} \eta(r,s)\delta_0(B_r^1 - B_s^2 - y)f(y)dydrds.$$

This notation emphasizes dependence on B^1, B^2, and the formal interpretation of $L_t(\eta)$ as a "convoluted intersection local time" of B^1 and B^2.

Proof of Lemma 4.6: As in [18], the proof follows by classical methods. We include it for the sake of completeness. To simplify the writing, we omit η in the arguments below.

a) Note that $E(L_{t,\varepsilon}L_{t,\delta}) = \int_{[0,t]^4} \eta(s_1,t_1)\eta(s_2,t_2)\psi_{\varepsilon,\delta}^{(2)}(\mathbf{s},\mathbf{t})d\mathbf{s}d\mathbf{t}$, where

$$\begin{aligned}\psi_{\varepsilon,\delta}^{(2)}(\mathbf{s},\mathbf{t}) \quad &:= \quad E\left[\int_{(\mathbb{R}^d)^2} p_\varepsilon(B_{s_1}^1 - B_{t_1}^2 - y_1)p_\varepsilon(B_{s_2}^1 - B_{t_2}^2 - y_2)f(y_1)f(y_2)dy_1dy_2\right] \\ &= \quad (2\pi)^{-2d}\int_{(\mathbb{R}^d)^2} \exp\left\{-\frac{1}{2}\sum_{j,k=1}^2 \sigma_{jk}\xi_j\cdot\xi_k - \frac{\varepsilon}{2}|\xi_1|^2 - \frac{\delta}{2}|\xi_2|^2\right\}\mu(d\xi_1)\mu(d\xi_2).\end{aligned}$$

(The second equality above can be proved using the same argument as in the proof of Lemma 4.3.) Then $\lim_{\varepsilon,\delta\downarrow 0}\psi_{\varepsilon,\delta}^{(2)}(\mathbf{s},\mathbf{t}) = \psi^{(2)}(\mathbf{s},\mathbf{t})$. Relation (26) follows by the dominated convergence theorem, since $\psi_{\varepsilon,\delta}^{(2)}(\mathbf{s},\mathbf{t}) \le \psi^{(2)}(\mathbf{s},\mathbf{t})$ for all $\varepsilon,\delta > 0$, and

$$\int_{[0,t]^4} \eta(s_1,t_1)\eta(s_2,t_2)\psi^{(2)}(\mathbf{s},\mathbf{t})d\mathbf{s}d\mathbf{t} = \alpha_2(t) < \infty.$$

¿From here, we also infer that $\lim_{\varepsilon\downarrow 0} E(L_{t,\varepsilon}^2)=\alpha_2(t)$, and hence

$$\lim_{\varepsilon,\delta\downarrow 0} E|L_{t,\varepsilon}-L_{t,\delta}|^2=\lim_{\varepsilon\downarrow 0} E(L_{t,\varepsilon}^2)+\lim_{\delta\downarrow 0} E(L_{t,\delta}^2)-2\lim_{\varepsilon,\delta\downarrow 0} E(L_{t,\varepsilon}L_{t,\delta})=0. \quad (28)$$

Let $(\varepsilon_n)_n\downarrow 0$ be arbitrary. From (28), it follows that $(L_{t,\varepsilon_n})_n$ is a Cauchy sequence in $L^2(\Omega)$. Hence, there exists $L_t\in L^2(\Omega)$ such that $E|L_{t,\varepsilon_n}-L_t|^2\to 0$. If $(\varepsilon_n')_n\downarrow 0$ is another sequence and $E|L_{t,\varepsilon_n'}-L_t'|^2\to 0$ for some $L_t'\in L^2(\Omega)$, then $E|L_t-L_t'|^2\le E|L_t-L_{t,\varepsilon_n}|^2+E|L_{t,\varepsilon_n}-L_{t,\varepsilon_n'}|^2+E|L_{t,\varepsilon_n'}-L_t'|^2\to 0$, i.e. $E|L_t-L_t'|^2=0$. This shows that L_t does not depend on $(\varepsilon_n)_n$.

b) Let $p\ge 2$ be fixed. Let $(\varepsilon_n)_n\downarrow 0$ be arbitrary. We will prove that $E|L_{t,\varepsilon_n}-L_t|^p\to 0$, by using the fact that, in a metric space, $x_n\to x$ if and only if for any subsequence $N'\subset\mathbb{N}$ there exists a sub-subsequence $N''\subset N'$ such that $x_n\to x$, as $n\to\infty, n\in N''$ (see e.g. p.15 of [4]).

Let $N'\subset\mathbb{N}$ be an arbitrary subsequence. By part a), as $n\to\infty, n\in N'$, $L_{t,\varepsilon_n}\to L_t$ in $L^2(\Omega)$. Hence, $L_{t,\varepsilon_n}\to L_t$ in probability, and there exists a sub-subsequence $N''\subset N'$ such that $L_{t,\varepsilon_n}\to L_t$ a.s., as $n\to\infty, n\in N''$. Note that $(L_{t,\varepsilon})_{\varepsilon>0}$ is uniformly integrable, since

$$\sup_{\varepsilon>0} E(L_{t,\varepsilon}^n)=\sup_{\varepsilon>0}\alpha_{n,\varepsilon}(t)\le\alpha_n(t)<\infty, \quad \text{for } n\ge 2.$$

By Theorem 16.14 of [5], it follows that $|L_t|^p$ is integrable and $E|L_{t,\varepsilon_n}-L_t|^p\to 0$, as $n\to\infty, n\in N''$. □

The next two results are the analogues of Propositions 3.1 and 3.2 of [18] in the case of a colored noise. We denote $\Phi(x,a)=\sum_{n=0}^{\infty} x^n/(n!)^a$ for $x>0$ and $a\ge 0$. Note that $\Phi(x,a)<\infty$ if and only if $a>0, x>0$ or $a=0, x\in(0,1)$.

Proposition 4.7 *Suppose that $\eta:[0,t]^2\to\mathbb{R}_+$ satisfies the following condition:*

$$\|\eta\|_{1,t}:=\max\left(\sup_{s\in[0,t]}\int_0^t\eta(r,s)dr,\ \sup_{r\in[0,t]}\int_0^t\eta(r,s)ds\right)<\infty. \quad (29)$$

(i) If f is the Riesz kernel of order α, or the Bessel kernel of order $\alpha<d$, and $d<2+\alpha$, then $\lim_{\varepsilon\downarrow 0} L_{t,\varepsilon}(\eta)=L_t(\eta)$ exists in $L^p(\Omega)$ for all $p\ge 2$, and

$$\sup_{\varepsilon>0} E\left[\exp\left(\lambda L_{t,\varepsilon}(\eta)\right)\right]\le C_{\alpha,d}^*\Phi\left(\lambda D(t),1-\frac{d-\alpha}{2}\right), \quad \textit{for all } \lambda>0,$$

where $C_{\alpha,d}^*$ is a constant depending on α and d, and

$$D(t)=D_{\alpha,d}2^{-(d-\alpha)/2}\|\eta\|_{1,t}\Gamma\left(1-\frac{d-\alpha}{2}\right)\left(1-\frac{d-\alpha}{2}\right)^{-[1-(d-\alpha)/2]}t^{1-(d-\alpha)/2}.$$

(ii) If f is the heat kernel of order α, or the Poisson kernel of order α, then $\lim_{\varepsilon\downarrow 0} L_{t,\varepsilon}(\eta)=L_t(\eta)$ exists in $L^p(\Omega)$, for all $p\ge 2$, and

$$\sup_{\varepsilon>0} E\left[\exp\left(\lambda L_{t,\varepsilon}(\eta)\right)\right]\le\exp(\lambda C(t)), \quad \textit{for all } \lambda>0,$$

where $C(t)=C_{\alpha,d}\|\eta\|_{1,t}t$.

Proof: We use the definition (25) of $\alpha_n(t,\eta)$, and Lemma 3.4 for the estimation of $\psi^{*(n)}(\mathbf{s},\mathbf{t})$.

(i) Let $h=-(d-\alpha)/2$. Using the Cauchy-Schwartz inequality, condition (29), and Lemma 3.5, we get:

$$\begin{aligned}
\alpha_n(t,\eta) &\le \left(D_{\alpha,d}2^{-(d-\alpha)/2}\right)^n \int_{[0,t]^{2n}} \prod_{j=1}^n \eta(s_j,t_j)[\beta(\mathbf{s})\beta(\mathbf{t})]^{-(d-\alpha)/4} d\mathbf{s}d\mathbf{t} \\
&\le \left(D_{\alpha,d}2^{-(d-\alpha)/2}\right)^n \int_{[0,t]^{2n}} \prod_{j=1}^n \eta(s_j,t_j)[\beta(\mathbf{s})]^{-(d-\alpha)/2} d\mathbf{s}d\mathbf{t} \\
&\le \left(D_{\alpha,d}2^{-(d-\alpha)/2}\right)^n \|\eta\|_{1,t}^n \int_{[0,t]^n} [\beta(\mathbf{s})]^{-(d-\alpha)/2} d\mathbf{s} \\
&= \left(D_{\alpha,d}2^{-(d-\alpha)/2}\|\eta\|_{1,t}\right)^n n!\; I_n(t,h) \\
&= \Gamma(1+h)\left(D_{\alpha,d}2^{-(d-\alpha)/2}\|\eta\|_{1,t}\Gamma(1+h)t^{1+h}\right)^n \frac{n!}{\Gamma(n(1+h)+1)} \\
&\le C^*_{\alpha,d}D(t)^n(n!)^{-h}.
\end{aligned}$$

(The last inequality follows by relation (3.19) of [18].) The first statement follows by Lemma 4.6. The second statement follows since,

$$E[e^{\lambda L_{t,\varepsilon}(\eta)}] = \sum_{n=0}^\infty \frac{\lambda^n}{n!}\alpha_{n,\varepsilon}(t,\eta) \le \sum_{n=0}^\infty \frac{\lambda^n}{n!}\alpha_n(t,\eta) \le C^*_{\alpha,d}\sum_{n=0}^\infty \frac{[\lambda D(t)]^n}{(n!)^{1+h}},$$

and the last sum is finite for all $\lambda>0$, since $1+h>0$.

(ii) In this case,

$$\alpha_n(t,\eta) \quad \le C^n_{\alpha,d}\int_{[0,t]^{2n}}\prod_{j=1}^n \eta(s_j,t_j)d\mathbf{s}d\mathbf{t} \le C^n_{\alpha,d}\|\eta\|^n_{1,t}t^n = C(t)^n,$$

and

$$E[e^{\lambda L_{t,\varepsilon}(\eta)}] \le \sum_{n=0}^\infty \frac{\lambda^n}{n!}\alpha_n(t,\eta) \le \sum_{n=0}^\infty \frac{[\lambda C(t)]^n}{n!} = e^{\lambda C(t)}.$$

□

Proposition 4.8 *Suppose that $\eta:[0,t]^2\to\mathbb{R}_+$ satisfies the following condition: there exist $\gamma>0$ and $1/2<H<1$, such that*

$$\eta(r,s)\le\gamma|r-s|^{2H-2}, \quad \forall r,s\in[0,t]. \tag{30}$$

(i) If f is the Riesz kernel of order α, or f is the Bessel kernel of order $\alpha<d$, and $d\le 2+\alpha$, then $\lim_{\varepsilon\downarrow 0}L_{t,\varepsilon}(\eta)=L_t(\eta)$ exists in $L^p(\Omega)$, for all $p\ge 2$, and

$$\sup_{\varepsilon>0}E\left[\exp\left(\lambda L_{t,\varepsilon}(\eta)\right)\right] \le C^*_{H,d,\alpha}\Phi\left(\lambda D(t), 1-\frac{d-\alpha}{2}\right), \quad \textit{for all } 0<\lambda<\lambda_0(t),$$

where $C^*_{H,d,\alpha}$ is a constant depending on H, d and α,

$$D(t) = D_{\alpha,d}2^{-(d-\alpha)/2}\frac{\gamma}{\alpha_H}\beta_H^2\Gamma\left(1-\frac{d-\alpha}{4H}\right)^{2H}\left(1-\frac{d-\alpha}{4H}\right)^{-[2H-(d-\alpha)/(2)]}t^{2H-(d-\alpha)/2}$$

and

$$\lambda_0(t) = \begin{cases} \left(1-\frac{1}{2H}\right)^{2H-1} D_{\alpha,d}^{-1}2\gamma^{-1}\beta_H^{-2}\Gamma\left(1-\frac{1}{2H}\right)^{-2H}t^{1-2H} & \text{if } d = 2+\alpha \\ \infty & \text{if } d < 2+\alpha \end{cases}$$

(ii) If f is the heat kernel of order α or the Poisson kernel of order α, then then $\lim_{\varepsilon\downarrow 0} L_{t,\varepsilon}(\eta) = L_t(\eta)$ exists in $L^p(\Omega)$, for all $p \geq 2$, and

$$\sup_{\varepsilon>0} E\left[\exp\left(\lambda L_{t,\varepsilon}(\eta)\right)\right] \leq \exp(\lambda C(t)), \quad \text{for all } \lambda > 0,$$

where $C(t) = C_{\alpha,d}\gamma t^{2H}/\alpha_H$.

Proof: The proof is similarly to Proposition 3.6. We use the definition (25) of $\alpha_n(t,\eta)$, Lemma 3.4 and condition (30).

(i) We have:

$$\alpha_n(t,\eta) \leq C^*_{H,d,\alpha}D(t)^n\ (n!)^{(d-\alpha)/2} < \infty.$$

The first statement follows by Lemma 4.6. The other statement follows, since

$$E[e^{\lambda L_{t,\varepsilon}(\eta)}] = \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\alpha_{n,\varepsilon}(t,\eta) \leq \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\alpha_n(t,\eta) \leq C^*_{H,d,\alpha}\sum_{n=0}^{\infty}\frac{[\lambda D(t)]^n}{(n!)^{1-(d-\alpha)/2}}.$$

If $d-\alpha = 2$, then the last sum is finite for all $0 < \lambda < \lambda_0(t) := 1/D(t)$. If $d-\alpha < 2$, then then last sum is finite for all $\lambda > 0$.

(ii) The result follows, since:

$$\begin{aligned} \alpha_n(t,\eta) &\leq C^n_{\alpha,d}\int_{[0,t]^{2n}}\prod_{j=1}^n \eta(s_j,t_j)d\mathbf{s}d\mathbf{t} \leq C^n_{\alpha,d}\gamma^n\int_{[0,t]^{2n}}\prod_{j=1}^n |s_j-t_j|^{2H-2}d\mathbf{s}d\mathbf{t} \\ &= \left(C_{\alpha,d}\frac{\gamma}{\alpha_H}t^{2H}\right)^n = C(t)^n \end{aligned}$$

and hence

$$E[e^{\lambda L_{t,\varepsilon}(\eta)}] \leq \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\alpha_n(t,\eta) \leq \sum_{n=0}^{\infty}\frac{[\lambda C(t)]^n}{n!}.$$

□

We now introduce the approximation technique of [18], which will yield simultaneously the existence of the solution of (1) and some representation formulas for the moments of this solution. We review briefly this powerful technique, which has been introduced only recently in the literature. The idea is to smooth the noise $\dot{W}$, solve the equation driven by the smoothen noise, and then show that the solution of the "smoothen" equation converges to the solution of (1).

For any $\varepsilon, \delta > 0$, let $\varphi_\delta(t) = \delta^{-1} 1_{[0,\delta]}(t)$ and

$$\dot{W}^{\varepsilon,\delta}_{t,x} = \int_0^t \int_{\mathbb{R}^d} \varphi_\delta(t-s) p_\varepsilon(x-y) dW_{s,y}.$$

Note that the noise $\dot{W}^{\varepsilon,\delta}$ can be viewed as a "mollification" of $\dot{W}$, with rate δ in the time variable and rate $\sqrt{\varepsilon}$ is the space variable, since

$$\varphi_\delta = \frac{1}{\delta}\varphi\left(\frac{t}{\delta}\right) \quad \text{and} \quad p_\varepsilon(x) = \frac{1}{(\sqrt{\varepsilon})^d}\phi\left(\frac{x}{\sqrt{\varepsilon}}\right),$$

with $\varphi(t) = 1_{[0,1]}(t)$ and $\phi(x) = (2\pi)^{-d/2} e^{-|x|^2/2}$. (Recall that the function $u^{(\varepsilon)}$, defined by $u^{(\varepsilon)}(x) = \int_{\mathbb{R}^n} \psi_\varepsilon(x-y)u(y)dy$, is a "mollification" of the function u on $\mathbb{R}^n$, if $\psi_\varepsilon(x) = \varepsilon^{-n}\psi(x/\varepsilon)$ and $\psi \geq 0$ is such that $\int_{\mathbb{R}^n} \psi(x)dx = 1$.) Therefore, this approximation procedure can be regarded as a stochastic version of the "approximation to the identity" technique, encountered in the PDE literature.

We consider the following "approximation" of equation (1):

$$\begin{aligned} \frac{\partial u^{\varepsilon,\delta}}{\partial t} &= \frac{1}{2}\Delta u^{\varepsilon,\delta} + u^{\varepsilon,\delta}\dot{W}^{\varepsilon,\delta}, \quad t > 0, x \in \mathbb{R}^d \\ u^{\varepsilon,\delta}_{0,x} &= u_0(x), \quad x \in \mathbb{R}^d. \end{aligned} \tag{31}$$

We introduce now the rigorous meaning for the solution of (31), which could be derived formally from the mild or evolution version of the equation, by applying the stochastic Fubini theorem.

Definition 4.9 *An $(\mathcal{F}_t)_t$-adapted square-integrable process $u = \{u^{\varepsilon,\delta}_{t,x}; (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ is a* **solution to** *(31) if for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, the process*

$$\left\{Y^{t,x,\varepsilon,\delta}_{r,z} = 1_{[0,t]}(r) \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)\varphi_\delta(s-r)p_\varepsilon(y-z)u^{\varepsilon,\delta}_{s,y} dyds;\ (r,z) \in \mathbb{R}_+ \times \mathbb{R}^d\right\}$$

exists, is integrable with respect to W, and satisfies

$$u^{\varepsilon,\delta}_{t,x} = p_t u_0(x) + \int_0^\infty \int_{\mathbb{R}^d} Y^{t,x,\varepsilon,\delta}_{r,z} \delta W_{r,z}.$$

By (7), the above definition is equivalent to saying that for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, the process $Y^{t,x,\varepsilon,\delta}$ exists, $u^{\varepsilon,\delta}_{t,x} \in L^2(\Omega)$, $u^{\varepsilon,\delta}_{t,x}$ is $\mathcal{F}_t$-measurable and

$$E(u^{\varepsilon,\delta}_{t,x} F) = E(F)p_t u_0(x) + E\langle Y^{t,x,\varepsilon,\delta}, DF\rangle_{\mathcal{HP}}, \quad \forall F \in \mathbb{D}^{1,2}. \tag{32}$$

Before constructing the solution of (31), we mention few words about the notation. If X and Y are random variables defined on $(\Omega, \mathcal{F}, P)$, with values in arbitrary measurable spaces $\mathcal{X}$, respectively $\mathcal{Y}$, and $h : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$

is a measurable function, we define the random variable: $E^X[h(X,Y)](\omega) = \int_{\mathcal{X}} h(x,Y(\omega))(P\circ X^{-1})(dx)$. If X and Y are independent, then

$$E[E^X[h(X,Y)]] = E[h(X,Y)] = E[E[h(X,Y)|X]], \tag{33}$$

where $E[\,\cdot\,]$ denotes the expectation with respect to P, and $E[\,\cdot\,|X]$ denotes the conditional expectation given X. (This result will be used below with $X = B$ and $Y = W$.)

We have the following result.

Proposition 4.10 *The process $u^{\varepsilon,\delta} = \{u^{\varepsilon,\delta}_{t,x}; (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ defined by:*

$$u^{\varepsilon,\delta}_{t,x} := E^B\left[u_0(x+B_t)\exp\left(\int_0^t \int_{\mathbb{R}^d} A^{\varepsilon,\delta,B}_{r,y} dW_{r,y} - \frac{1}{2}\|A^{\varepsilon,\delta,B}\|^2_{\mathcal{HP}}\right)\right] \tag{34}$$

is a solution of (31), where $A^{\varepsilon,\delta,B}_{r,y} = \int_0^t \varphi_\delta(t-s-r)p_\varepsilon(x+B_s-y)ds$, and $B = (B_t)_{t\geq 0}$ is a d-dimensional standard Brownian motion, independent of W.

Proof: The argument is similar to the one used in the proof of Proposition 5.2 of [18]. We include it in response to the referee's suggestion. To simplify the notation, we omit writing $\mathcal{HP}$ in $\|\cdot\|_{\mathcal{HP}}$ and $\langle\cdot,\cdot\rangle_{\mathcal{HP}}$. We also omit writing ε,δ in $A^{\varepsilon,\delta,B}$, i.e. we denote $A^{\varepsilon,\delta,B}$ by A^B.

For every $\varphi \in \mathcal{HP}$, define $F_\varphi = e^{W(\varphi)-\|\varphi\|^2/2}$. Note that

$$E(e^{W(\varphi)}) = e^{\|\varphi\|^2/2}, \quad \forall \varphi \in \mathcal{HP}. \tag{35}$$

Since $\{F_\varphi; \varphi \in \mathcal{HP}\}$ is dense in $\mathbb{D}^{1,2}$ (see e.g. Lemma 1.1.2 of [28]), it suffices to prove (32) for $F = F_\varphi$. Define $S_{t,x}(\varphi) = E(u^{\varepsilon,\delta}_{t,x}F_\varphi)$. Using (33),

$$\begin{aligned}
S_{t,x}(\varphi) &= E[E^B[u_0(x+B_t)e^{W(A^B)-\|A^B\|^2/2}]e^{W(\varphi)-\|\varphi\|^2/2}] \\
&= E[E^B[u_0(x+B_t)e^{W(A^B+\varphi)-\|A^B+\varphi\|^2/2}e^{\langle A^B,\varphi\rangle}]] \\
&= E[E[u_0(x+B_t)e^{W(A^B+\varphi)-\|A^B+\varphi\|^2/2}e^{\langle A^B,\varphi\rangle}|B]].
\end{aligned}$$

Let

$$h(B,W) = u_0(x+B_t)e^{W(A^B+\varphi)-\|A^B+\varphi\|^2/2}e^{\langle A^B,\varphi\rangle}.$$

Since B and W are independent, $E[h(B,W)|B] = f(B)$, where

$$\begin{aligned}
f(b) &= E[h(b,W)] = E[u_0(x+b_t)e^{W(A^b+\varphi)-\|A^b+\varphi\|^2/2}e^{\langle A^b,\varphi\rangle}] \\
&= u_0(x+b_t)e^{\langle A^b,\varphi\rangle}E[e^{W(A^b+\varphi)-\|A^b+\varphi\|^2/2}] \\
&= u_0(x+b_t)e^{\langle A^b,\varphi\rangle}, \quad \text{for any } b = (b_t)_{t\geq 0} \in C([0,\infty),\mathbb{R}^d),
\end{aligned}$$

where $C([0,\infty),\mathbb{R}^d)$ denotes the space of continuous functions $x : [0,\infty) \to \mathbb{R}^d$, and we used (35) for the last equality. Hence $E[h(B,W)|B] = u_0(x+B_t)e^{\langle A^B,\varphi\rangle}$ and

$$S_{t,x}(\varphi) = E[E[h(B,W)|B]] = E[u_0(x+B_t)e^{\langle A^B,\varphi\rangle}].$$

By the definition of A^B and Fubini's theorem, we obtain:

$$\begin{aligned}\langle A^B,\varphi\rangle &= \alpha_H\int_{(\mathbb{R}_+\times\mathbb{R}^d)^2} A^B_{r,y}\varphi_{r',y'}|r-r'|^{2H-2}f(y-y')dydy'drdr'\\ &= \int_0^t V^{\varepsilon,\delta}(t-s,x+B_s)ds,\end{aligned}$$

where $V^{\varepsilon,\delta}(t,x)=\langle\varphi_\delta(t-\cdot)p_\varepsilon(x-\cdot),\varphi\rangle$. Hence:

$$S_{t,x}(\varphi)=E\left[u_0(x+B_t)\exp\left(\int_0^t V^{\varepsilon,\delta}(t-s,x+B_s)ds\right)\right].$$

By the Feynman-Kac's formula (see e.g. Theorem 5.7.6 of [19]), $(S_{t,x}(\varphi))_{t,x}$ is a solution of the Cauchy problem:

$$\begin{aligned}\frac{\partial S_{t,x}(\varphi)}{\partial t} &= \frac{1}{2}\Delta S_{t,x}(\varphi)+S_{t,x}(\varphi)V^{\varepsilon,\delta}(t,x), \quad t>0, x\in\mathbb{R}^d\\ S_{0,x}(\varphi) &= u_0(x).\end{aligned}$$

Hence,

$$\begin{aligned}S_{t,x}(\varphi) &= p_y u_0(x)+\int_0^t\int_{\mathbb{R}^d} p_{t-s}(x-y)S_{s,y}(\varphi)V^{\varepsilon,\delta}(s,y)dyds\\ &= p_y u_0(x)+\alpha_H E\int_{(\mathbb{R}_+\times\mathbb{R}^d)^2} Y^{t,x,\varepsilon,\delta}_{r,z}\varphi(r',z')F_\varphi|r-r'|^{2H-2}f(z-z')dzdz'drdr'\\ &= p_y u_0(x)+E\langle Y^{t,x,\varepsilon,\delta},DF_\varphi\rangle, \qquad (36)\end{aligned}$$

where we used Fubini's theorem for the second equality above and the fact that $D_{r',z'}F_\varphi=\varphi(r',z')F_\varphi$ for the third equality. This concludes the proof of (32) for $F=F_\varphi$. $\square$

Let $B^i=(B^i_t)_{t\geq 0}, i\geq 1$ be independent d-dimensional standard Brownian motions, independent of W. Suppose that either (18) or (19) hold. For any pair (i,j) with $i\neq j$, let $L_t^{B^i,B^j}$ be the random variable defined in Lemma 4.6, with

$$\eta(r,s)=\begin{cases}\alpha_H|r-s|^{2H-2} & \text{if } H>1/2\\ 1_{\{r=s\}} & \text{if } H=1/2\end{cases}$$

(By Proposition 3.6, $\alpha_n(t,\eta)<\infty$ for all $n\geq 1$, and $L_t^{B^i,B^j}$ is well-defined.)

The following result is the main theorem of the present article.

Theorem 4.11 *(i) Suppose that f is the Riesz kernel of order α or the Bessel kernel of order $\alpha<d$, and either (18) or (19) holds. Then, for any integer $k\geq 2$, we have:*

$$\sup_{\varepsilon,\delta>0} E[(u^{\varepsilon,\delta}_{t,x})^k]<\infty, \quad \textit{for all } 0<t<t_0(k), x\in\mathbb{R}^d \qquad (37)$$

where

$$t_0(k) = \begin{cases} \left[k(k-1)D_{\alpha,d}2^{-2H}\beta_H^2\Gamma\left(1-\frac{1}{2H}\right)^{2H}\right]^{-1/(2H-1)} & \text{if } d = 2+\alpha \\ \infty & \text{if } d < 2+\alpha \end{cases}$$

For any $0 < t < t_0(2)$ and $x \in \mathbb{R}^d$, the limit $u_{t,x} := \lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0} u_{t,x}^{\varepsilon,\delta}$ exists in $L^2(\Omega)$, the process $u = \{u_{t,x}; (t,x) \in [0, t_0(2)) \times \mathbb{R}^d\}$ is the unique solution of (1) in $L^2(\Omega)$, and

$$E[u_{t,x}^2] = E\left[u_0(x+B_t^1)u_0(x+B_t^2)\exp\left(L_t^{B^1,B^2}\right)\right] := \gamma_2(t,x).$$

If $x \in \mathbb{R}^d$ and $t < t_0(M)$ for some $M \geq 3$, then $\lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0} E|u_{t,x}^{\varepsilon,\delta} - u_{t,x}|^p = 0$ for all $2 \leq p < M$, and for any integer $2 \leq k \leq M-1$,

$$E[u_{t,x}^k] = E\left[\prod_{j=1}^k u_0(x+B_t^i)\exp\left(\sum_{1\leq i<j\leq k} L_t^{B^i,B^j}\right)\right] := \gamma_k(t,x). \quad (38)$$

(ii) Suppose that f is the heat kernel of order α, or the Poisson kernel of order α. Then the conclusion same as in part (i) holds, with $t_0(k) = \infty$ for all $k \geq 2$.

Proof: The argument is similar to the one used in the proof of Theorem 5.3 of [18]. At the referee's request, we include all the details for the reader's convenience. To ease the exposition, we divide the proof in several steps.

Step 1. We show that for any integer $k \geq 2$,

$$E[(u_{t,x}^{\varepsilon,\delta})^k] = E\left[\prod_{j=1}^k u_0(x+B_t^j)\exp\left(\sum_{1\leq i<j\leq k}\langle A^{\varepsilon,\delta,B^i}, A^{\varepsilon,\delta,B^j}\rangle_{\mathcal{HP}}\right)\right]. \quad (39)$$

By (34), $u_{t,x}^{\varepsilon,\delta}$ can be expressed as

$$u_{t,x}^{\varepsilon,\delta} = E^{B^i}\left[u_0(x+B_t^i)\exp\left(\int_0^t\int_{\mathbb{R}^d} A_{r,y}^{\varepsilon,\delta,B^i}dW_{r,y} - \frac{1}{2}\|A^{\varepsilon,\delta,B^i}\|_{\mathcal{HP}}^2\right)\right],$$

for any $i = 1,\ldots,k$. Taking the product over $i = 1,\ldots,k$ and using the independence of $B^1,\ldots,B^k$, we obtain that:

$$\begin{aligned}(u_{t,x}^{\varepsilon,\delta})^k &= \prod_{i=1}^k E^{B^i}\left[u_0(x+B_t^i)\exp\left(\int_0^t\int_{\mathbb{R}^d} A_{r,y}^{\varepsilon,\delta,B^i}dW_{r,y} - \frac{1}{2}\|A^{\varepsilon,\delta,B^i}\|_{\mathcal{HP}}^2\right)\right] \\ &= E^{B^1,\ldots B^k}\left[\prod_{i=1}^k u_0(x+B_t^i)\exp\left(\int_0^t\int_{\mathbb{R}^d} A_{r,y}^{\varepsilon,\delta,B^i}dW_{r,y} - \frac{1}{2}\|A^{\varepsilon,\delta,B^i}\|_{\mathcal{HP}}^2\right)\right]\end{aligned}$$

Taking the expectation, and using (33) with $X=(B^1,\ldots,B^k):=B$ and $Y=W$, we get:

$$E[(u^{\varepsilon,\delta}_{t,x})^k] \;=\; E\left[E\left[\prod_{i=1}^k u_0(x+B^i_t)\exp\left(\int_0^t\int_{\mathbb{R}^d} A^{\varepsilon,\delta,B^i}_{r,y}dW_{r,y}-\frac{1}{2}\|A^{\varepsilon,\delta,B^i}\|^2_{\mathcal{HP}}\right)|B\right]\right].$$

Let

$$h(B,W)=\prod_{i=1}^k u_0(x+B^i_t)\exp\left(\int_0^t\int_{\mathbb{R}^d} A^{\varepsilon,\delta,B^i}_{r,y}dW_{r,y}-\frac{1}{2}\|A^{\varepsilon,\delta,B^i}\|^2_{\mathcal{HP}}\right).$$

Then $E[h(B,W)|B]=f(B)$, where

$$\begin{aligned}
f(b) &= E[h(b,W)]=E\left[\prod_{i=1}^k u_0(x+b^i_t)e^{W(A^{\varepsilon,\delta,b^i})-\|A^{\varepsilon,\delta,b^i}\|^2_{\mathcal{HP}}/2}\right]\\
&= \prod_{i=1}^k u_0(x+b^i_t)e^{-\sum_{i=1}^k\|A^{\varepsilon,\delta,b^i}\|^2_{\mathcal{HP}}/2}E\left[e^{W\left(\sum_{i=1}^k A^{\varepsilon,\delta,b^i}\right)}\right]\\
&= \prod_{i=1}^k u_0(x+b^i_t)\exp\left(-\frac{1}{2}\sum_{i=1}^k\|A^{\varepsilon,\delta,b^i}\|^2_{\mathcal{HP}}+\frac{1}{2}\left\|\sum_{i=1}^k A^{\varepsilon,\delta,b^i}\right\|^2_{\mathcal{HP}}\right)\\
&= \prod_{i=1}^k u_0(x+b^i_t)\exp\left(\sum_{i<j}\langle A^{\varepsilon,\delta,b^i},A^{\varepsilon,\delta,b^j}\rangle_{\mathcal{HP}}\right)
\end{aligned}$$

for any $b=(b^1,\ldots,b^k)$ with $b^i=(b^i_t)_{t\geq 0}\in C([0,\infty),\mathbb{R}^d)$. (We used (35) and the fact that $W(\varphi+\psi)=W(\varphi)+W(\psi)$ a.s. for any $\varphi,\psi\in\mathcal{HP}$, which can be checked in $L^2(\Omega)$, using the fact that W is an isometry between $\mathcal{HP}$ and $L^2(\Omega)$.) Relation (39) follows, since $E[(u^{\varepsilon,\delta}_{t,x})^k]=E[E[h(B,W)|B]]=E[f(B)]$.

Step 2. We prove that for any $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$ with $t<t_0(2)$,

$$\lim_{\delta\downarrow 0}\langle A^{\varepsilon,\delta,B^i},A^{\varepsilon,\delta,B^j}\rangle_{\mathcal{HP}}=L^{B^i,B^j}_{t,2\varepsilon},\quad \forall\varepsilon>0,\quad \forall\omega\in\tilde{\Omega}_{i,j},\tag{40}$$

where $\tilde{\Omega}_{i,j}=\{\omega\in\Omega;\ B^i(\omega)\text{ and }B^j(\omega)\text{ are continuous}\}$ $(P(\tilde{\Omega}_{i,j})=1)$.

Let $\omega\in\tilde{\Omega}_{i,j}$ and $\varepsilon>0$ fixed. For any $(s_1,s_2)\in[0,t]^2$, define

$$\eta_\delta(s_1,s_2)=\begin{cases}\alpha_H\int_0^t\int_0^t\varphi_\delta(t-s_1-r_1)\varphi_\delta(t-s_2-r_2)|r_1-r_2|^{2H-2}dr_1dr_2 & \text{if } H>1/2\\ \int_0^t\varphi_\delta(t-s_1-r)\varphi_\delta(t-s_2-r)dr & \text{if } H=1/2\end{cases}.$$

By direct calculation, using Fubini's theorem and the fact that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}p_\varepsilon(x+B^i_{s_1}-y_1)p_\varepsilon(x+B^j_{s_2}-y_2)f(y_1-y_2)dy_2dy_1=\int_{\mathbb{R}^d}p_{2\varepsilon}(B^i_{s_1}-B^j_{s_2}-y)f(y)dy,$$

(which can be proved by observing that $p_{\varepsilon_1} * p_{\varepsilon_2} = p_{\varepsilon_1+\varepsilon_2}$), it follows that

$$\begin{aligned}\langle A^{\varepsilon,\delta,B^i}, A^{\varepsilon,\delta,B^j}\rangle_{\mathcal{HP}} &= \int_0^t \int_0^t \int_{\mathbb{R}^d} \eta_\delta(s_1-s_2)p_{2\varepsilon}(B^i_{s_1}-B^j_{s_2}-y)f(y)dyds_1ds_2 \\ &= L^{B^i,B^j}_{t,2\varepsilon}(\eta_\delta). \end{aligned} \tag{41}$$

Note that, for any continuous function $g : [0,t]^2 \to \mathbb{R}$,

$$\lim_{\delta\downarrow 0} \int_0^t \int_0^t \eta_\delta(s_1,s_2)g(s_1,s_2)ds_1ds_2 = \begin{cases} \alpha_H \int_0^t \int_0^t |s_1-s_2|^{2H-2}g(s_1,s_2)ds_1ds_2 & \text{if } H>1/2 \\ \int_0^t g(s,s)ds & \text{if } H=1/2 \end{cases}$$

In particular, we consider the (random) function $g_{2\varepsilon}$ defined by: $g_{2\varepsilon}(s_1,s_2) = \int_{\mathbb{R}^d} p_{2\varepsilon}(B^i_{s_1}-B^j_{s_2}-y)f(y)dy = (p_{2\varepsilon}*f)(B^i_{s_1}-B^j_{s_2})$, for $(s_1,s_2)\in[0,t]^2$. (Note that $g_{2\varepsilon}$ is continuous by Remark 4.4.) Then,

$$\langle A^{\varepsilon,\delta,B^i}, A^{\varepsilon,\delta,B^j}\rangle_{\mathcal{HP}} = \int_0^t \int_0^t \eta_\delta(s_1,s_2)g_{2\varepsilon}(s_1,s_2)ds_1ds_2,$$

and

$$\begin{aligned}\lim_{\delta\downarrow 0}\langle A^{\varepsilon,\delta,B^i}, A^{\varepsilon,\delta,B^j}\rangle_{\mathcal{HP}} &= \begin{cases} \alpha_H \int_0^t \int_0^t |s_1-s_2|^{2H-2}g_{2\varepsilon}(s_1,s_2)ds_1ds_2 & \text{if } H>1/2 \\ \int_0^t g_{2\varepsilon}(s,s)ds & \text{if } H=1/2 \end{cases} \\ &= L^{B^i,B^j}_{t,2\varepsilon}. \end{aligned}$$

Step 3. We prove that for any $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$ with $t<t_0(2)$,

$$\{\exp(\langle A^{\varepsilon,\delta,B^i}, A^{\varepsilon,\delta,B^j}\rangle_{\mathcal{HP}})\}_{\varepsilon,\delta>0} \quad \text{is uniformly integrable,} \tag{42}$$

Suppose first that $H=1/2$. Then η_δ satisfies condition (29) with $\|\eta_\delta\|_{1,t}\le 1$ (see p. 318 of [18]). By applying Proposition 4.7 and using (41), it follows that if f is the Riesz or Bessel kernel,

$$\sup_{\varepsilon>0} E\left[\exp\left(\lambda\langle A^{\varepsilon,\delta,B^i}, A^{\varepsilon,\delta,B^j}\rangle_{\mathcal{HP}}\right)\right] \le C^*_{\alpha,d}\Phi\left(\lambda D(t), 1-\frac{d-\alpha}{2}\right), \quad \forall\lambda>0,$$

whereas if f is the heat or Poisson kernel,

$$\sup_{\varepsilon>0} E\left[\exp\left(\lambda\langle A^{\varepsilon,\delta,B^i}, A^{\varepsilon,\delta,B^j}\rangle_{\mathcal{HP}}\right)\right] \le e^{\lambda C(t)}, \quad \forall\lambda>0.$$

Note that both constants $D(t)$ and $C(t)$ depend (linearly) on $\|\eta_\delta\|_{1,t}$ (which is bounded by 1), and the function $\Phi(x,a)$ is increasing in x. We infer that there exists an upper bound for the above supremum over ε, which does not depend on δ. More precisely, denoting by $D(t), C(t)$ the respective constants $D(t), C(t)$, in which $\|\eta_\delta\|_{1,t}$ is replaced by 1, we infer that if f is the Riesz or the Bessel kernel,

$$\sup_{\varepsilon,\delta>0} E\left[\exp\left(\lambda\langle A^{\varepsilon,\delta,B^i}, A^{\varepsilon,\delta,B^j}\rangle_{\mathcal{HP}}\right)\right] \le C^*_{\alpha,d}\Phi\left(\lambda D(t), 1-\frac{d-\alpha}{2}\right), \quad \forall\lambda>0, \tag{43}$$

whereas if f is the heat or the Poisson kernel,

$$\sup_{\varepsilon,\delta>0} E\left[\exp\left(\lambda\langle A^{\varepsilon,\delta,B^i}, A^{\varepsilon,\delta,B^j}\rangle_{\mathcal{HP}}\right)\right] \leq e^{\lambda C(t)}, \quad \forall \lambda > 0. \tag{44}$$

Relation (42) follows by a well-known criterion (see p. 218 of [5]) by taking an arbitrary $\lambda > 1$.

Suppose now that $H > 1/2$. Then η_δ satisfies condition (30) (see relation (5.13) of [18]). By applying Proposition 4.8 and using (41), it follows that, if f is the Riesz or the Bessel kernel,

$$\sup_{\varepsilon>0} E\left[\exp\left(\lambda\langle A^{\varepsilon,\delta,B^i}, A^{\varepsilon,\delta,B^j}\rangle_{\mathcal{HP}}\right)\right] \leq C^*_{H,d,\alpha}\Phi\left(\lambda D(t), 1 - \frac{d-\alpha}{2}\right),$$

for all $0 < \lambda < \lambda_0(t)$, whereas if f is the heat or the Poisson kernel,

$$\sup_{\varepsilon>0} E\left[\exp\left(\lambda\langle A^{\varepsilon,\delta,B^i}, A^{\varepsilon,\delta,B^j}\rangle_{\mathcal{HP}}\right)\right] \leq e^{\lambda C(t)}, \quad \forall \lambda > 0.$$

The constants $D(t)$ and $C(t)$ depend on γ, which depends only on H. From here, we infer that, if f is the Riesz kernel or the Bessel kernel,

$$\sup_{\varepsilon,\delta>0} E\left[\exp\left(\lambda\langle A^{\varepsilon,\delta,B^i}, A^{\varepsilon,\delta,B^j}\rangle_{\mathcal{HP}}\right)\right] \leq C^*_{H,d,\alpha}\Phi\left(\lambda D(t), 1 - \frac{d-\alpha}{2}\right) < \infty, \tag{45}$$

for all $0 < \lambda < \lambda_0(t)$, whereas if f is the heat kernel of the Poisson kernel,

$$\sup_{\varepsilon,\delta>0} E\left[\exp\left(\lambda\langle A^{\varepsilon,\delta,B^i}, A^{\varepsilon,\delta,B^j}\rangle_{\mathcal{HP}}\right)\right] \leq e^{\lambda C(t)} < \infty, \quad \forall \lambda > 0. \tag{46}$$

Relation (42) follows as before, noting that $1 < \lambda_0(t)$ (since $t < t_0(2)$).

Note that (37) is obtained as a by-product of (39) and (43)-(46), since:

$$\sup_{\varepsilon,\delta>0} E[(u^{\varepsilon,\delta}_{t,x})^k] \leq \|u_0\|^k_\infty \sup_{\varepsilon,\delta>0} E\left[\exp\left(\frac{k(k-1)}{2}\langle A^{\varepsilon,\delta,B^1}, A^{\varepsilon,\delta,B^2}\rangle_{\mathcal{HP}}\right)\right], \tag{47}$$

and

$$\frac{k(k-1)}{2} < \lambda_0(t) \quad \text{if and only if} \quad t < t_0(k).$$

Moreover, using (43)-(47), and the fact that $D(t), C(t)$ are increasing functions of t and $\lambda_0(t)$ is a decreasing function of t, we conclude that, for any $0 < T < t_0(k)$ and for any $(t,x) \in [0,T] \times \mathbb{R}^d$,

$$\sup_{\varepsilon,\delta>0} E[(u^{\varepsilon,\delta}_{t,x})^k] \leq \|u_0\|^k_\infty C^*_{H,d,\alpha}\Phi\left(\frac{k(k-1)}{2}D(T), 1 - \frac{d-\alpha}{2}\right),$$

if f is the Riesz kernel or the Bessel kernel, and

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \sup_{\varepsilon,\delta>0} E[(u^{\varepsilon,\delta}_{t,x})^k] \leq \|u_0\|^k_\infty \exp\left\{\frac{k(k-1)}{2}C(T)\right\},$$

if f is the heat or the Bessel kernel. Hence, for any $0 < T < t_0(k)$,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \sup_{\varepsilon,\delta>0} E[(u^{\varepsilon,\delta}_{t,x})^k] \le K_T(k) < \infty, \tag{48}$$

where $K_T(k)$ is a constant depending on u_0, k, H, d, α and T.

Step 4. We prove that for any $0 < t < t_0(k)$,

$$\lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0} E[(u^{\varepsilon,\delta}_{t,x})^k] = \gamma_k(t,x). \tag{49}$$

First we note that, for any $\varepsilon > 0$

$$\lim_{\delta\downarrow 0} E[(u^{\varepsilon,\delta}_{t,x})^k] = E\left[\prod_{j=1}^k u_0(x+B^j_t)\exp\left(\sum_{1\le i<j\le k} L^{B^i,B^j}_{t,2\varepsilon}\right)\right] := H_{t,x,k}(\varepsilon).$$

(This follows by Theorem 16.14 of [5], using (39), (40), and (42)). Next, we show that:

$$\lim_{\varepsilon\downarrow 0} H_{t,x,k}(\varepsilon) = \gamma_k(t,x).$$

For this, let $(\varepsilon_n)\downarrow 0$ be an arbitrary sequence. We have to show that:

$$\lim_{n\to\infty} H_{t,x,k}(\varepsilon_n) = \gamma_k(t,x). \tag{50}$$

We use the fact that in a metric space, $x_n \to x$ if and only if for any subsequence $N' \subset \mathbb{N}$ there exists a sub-subsequence $N'' \subset N'$ such that $x_n \to x$, as $n \to \infty, n \in N''$. Let $N' \subset \mathbb{N}$ be an arbitrary subsequence.

By Lemma 4.6, the limit $L^{B^i,B^j}_t := \lim_{\varepsilon\downarrow 0} L^{B^i,B^j}_{t,2\varepsilon}$ exists in $L^2(\Omega)$. (In this lemma, we take $\eta(r,s) = \alpha_H|r-s|^{2H-2}$ if $H > 1/2$, and $\eta(r,s) = 1_{\{r=s\}}$ if $H = 1/2$.) Hence $L^{B^i,B^j}_{t,2\varepsilon_n} \to L^{B^i,B^j}_t$ in probability, as $n \to \infty, n \in N'$, and there exists a sub-subsequence $N'' \subset N'$ such that

$$L^{B^i,B^j}_{t,2\varepsilon_n} \to L^{B^i,B^j}_t \quad \text{a.s.} \quad \text{as } n\to\infty, n \in N''.$$

Note that $E[\exp(\lambda L^{B^i,B^i}_{t,2\varepsilon})] < \infty$ for all $\lambda > 0$, respectively for all $0 < \lambda < \lambda_0(t)$, with $\lambda_0(t) > 1$. (If $H = 1/2$, we use Proposition 4.7 with $\eta(r,s) = 1_{\{r=s\}}$. If $H > 1/2$, we use Proposition 4.8 with $\eta(r,s) = \alpha_H|r-s|^{2H-2}$.) Hence, $\{\exp(L^{B^i,B^j}_{t,2\varepsilon})\}_{\varepsilon>0}$ is uniform integrable. By Theorem 16.14 of [5],

$$H_{t,x,k}(\epsilon_n) \to \gamma_k(t,x), \quad \text{as} \quad n\to\infty, n\in N''.$$

Relation (50) follows using the above-mentioned subsequence criterion.

Step 5. We prove that for any $0 < t < t_0(2)$,

$$\lim_{\varepsilon,\varepsilon'\downarrow 0}\lim_{\delta,\delta'\downarrow 0} E[u^{\varepsilon,\delta}_{t,x} u^{\varepsilon',\delta'}_{t,x}] = \gamma_2(t,x). \tag{51}$$

Similarly to (39) and (40), one can prove that:

$$E[u^{\varepsilon,\delta}_{t,x}u^{\varepsilon',\delta'}_{t,x}] = E[u_0(x+B^1_t)u_0(x+B^2_t)\exp(\langle A^{\varepsilon,\delta,B^1}, A^{\varepsilon',\delta',B^2}\rangle_{\mathcal{HP}})],$$

and

$$\lim_{\delta,\delta'\downarrow 0}\langle A^{\varepsilon,\delta,B^1}, A^{\varepsilon',\delta',B^2}\rangle_{\mathcal{HP}} = L^{B^1,B^2}_{t,\varepsilon+\varepsilon'} \quad \forall \varepsilon>0,\ \forall\varepsilon'>0 \text{ a.s.}$$

Relation (51) follows using the same argument as in *Step 4* (based on (42)).

Step 6. We prove that for any $t<t_0(2)$ and $x\in\mathbb{R}^d$, the limit

$$u_{t,x} := \lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0} u^{\varepsilon,\delta}_{t,x} \text{ exists in } L^2(\Omega).$$

Let $t<t_0(2)$ and $x\in\mathbb{R}^d$ be fixed. From (51), we obtain that:

$$\lim_{\varepsilon,\varepsilon'\downarrow 0}\lim_{\delta,\delta'\downarrow 0} E|u^{\varepsilon,\delta}_{t,x}-u^{\varepsilon',\delta'}_{t,x}|^2 = 0. \tag{52}$$

Let $(\varepsilon_n)_n\downarrow 0$ and $(\delta_n)_n\downarrow 0$ be arbitrary. From (52), it follows that $(u^{\varepsilon_n,\delta_n}_{t,x})_n$ is a Cauchy sequence in $L^2(\Omega)$. Hence, there exists $u_{t,x}\in L^2(\Omega)$ such that $E|u^{\varepsilon_n,\delta_n}_{t,x}-u_{t,x}|^2\to 0$. The fact that $u_{t,x}$ does not depend on $(\varepsilon_n)_n$ and (δ_n) is proved by a standard argument (see for instance, the proof of Lemma 4.6).

Step 7. We now prove (38).

If $t<t_0(M)$ for some $M\geq 3$, then $\sup_{\varepsilon,\delta>0}E[(u^{\varepsilon,\delta}_{t,x})^M]<\infty$, and hence, $\{(u^{\varepsilon,\delta}_{t,x})^p\}_{\varepsilon,\delta>0}$ is uniformly integrable, for any $2\leq p<M$. Since $u^{\varepsilon,\delta}_{t,x}-u_{t,x}\to 0$ a.s. (along a subsequence), we conclude that $E|u^{\varepsilon,\delta}_{t,x}-u_{t,x}|^p\to 0$. In particular, $\lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0}E[(u^{\varepsilon,\delta}_{t,x})^k]=E[u^k_{t,x}]$ for any $2\leq k<M$, and (38) follows by (49).

Step 8. We prove that $u=\{u_{t,x};(t,x)\in[0,t_0(2))\times\mathbb{R}^d\}$ is a solution of (1).

Let $(u^{\varepsilon,\delta}_{t,x})_{t,x}$ be the solution of equation (31) (whose existence is guaranteed by Proposition 4.10). Let $(\varepsilon_n)\downarrow 0$ and $(\delta_n)\downarrow 0$ be arbitrary. By *Step 6*, $u^{\varepsilon_n,\delta_n}_{t,x}\to u_{t,x}$ in $L^2(\Omega)$, and hence $u^{\varepsilon_n,\delta_n}_{t,x}\to u_{t,x}$ a.s. along a subsequence $N'\subset\mathbb{N}$. Since $u^{\varepsilon_n,\delta_n}_{t,x}$ is $\mathcal{F}_t$-measurable, it follows that $u_{t,x}$ is $\mathcal{F}_t$-measurable.

Let $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$ be fixed with $t<t_0(2)$. We have to show that (8) holds. As in the proof of Proposition 4.10, it suffices to show that (8) holds for $F=F_\varphi$ with $\varphi\in\mathcal{HP}$. Moreover, it suffices to take $\varphi=1_{[0,a]}\phi$, with $a\in\mathbb{R}_+,\phi\in\mathcal{P}(\mathbb{R}^d)$, since the class of these functions is dense in $\mathcal{HP}$.

The idea is to take the limit in (32), as $\varepsilon\downarrow 0,\delta\downarrow 0$. On the left-hand side, $E(F_\varphi u^{\varepsilon,\delta}_{t,x})\to E(F_\varphi u_{t,x})$, since $E|u^{\varepsilon,\delta}_{t,x}-u_{t,x}|^2\to 0$. On the right-hand side of (32), we have: (see (36))

$$E\langle Y^{t,x,\varepsilon,\delta}, DF_\varphi\rangle_{\mathcal{HP}} \;=\; \int_0^t\int_{\mathbb{R}^d} p_{t-s}(x-y)E(u^{\varepsilon,\delta}_{s,y}F_\varphi)V^{\varepsilon,\delta}(s,y)dyds.$$

We have $\lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0}E(Fu^{\varepsilon,\delta}_{s,y})=E(F_\varphi u_{s,y})$, for all $(s,y)\in[0,t]\times\mathbb{R}^d$. Note that $V^{\varepsilon,\delta}=\psi_{\varepsilon,\delta}*H$, where $\psi_{\varepsilon,\delta}(s,y)=\varphi_\delta(s)p_\varepsilon(y)$ and

$$H(s,y) := \alpha_H\int_{\mathbb{R}_+}\int_{\mathbb{R}^d}\varphi(s',y')|s-s'|^{2H_2}f(y-y')dy'ds'.$$

Hence, $\lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0} V^{\varepsilon,\delta}(s,y)=H(s,y)$, for all $(s,y)\in[0,t]\times\mathbb{R}^d$. Therefore, by the dominated convergence theorem (whose application is justified below),

$$\lim_{\varepsilon\downarrow 0}\lim_{\delta\downarrow 0} E\langle Y^{t,x,\varepsilon_n,\delta_n}, DF_{\varphi}\rangle_{\mathcal{HP}}=\int_0^t\int_{\mathbb{R}^d} p_{t-s}(x-y)E(u_{s,y}F_{\varphi})H(s,y)dyds.$$

A direct calculation shows that the limit above coincides with $E\langle Y^{t,x}, DF_{\varphi}\rangle_{\mathcal{HP}}$ (using the fact that $D_{s,y}F_{\varphi}=\varphi(s,y)F_{\varphi}$).

It remains to justify the application of the dominated convergence theorem. Using the Cauchy-Schwartz inequality and (48), we have:

$$\sup_{(s,y)\in[0,t]\times\mathbb{R}^d}\sup_{\varepsilon,\delta>0}|E(u_{s,y}^{\varepsilon,\delta}F_{\varphi})|\leq\{K_t(2)\}^{1/2}\{E(F_{\varphi}^2)\}^{1/2}:=K_t^*. \tag{53}$$

Note that $V^{\varepsilon,\delta}(s,y)=J^{\delta}(s)I_{\varepsilon}(y)$, where

$$J_{\delta}(s)=\langle\varphi_{\delta}(s-\cdot),1_{[0,a]}\rangle_{\mathcal{H}}\quad\text{and}\quad I_{\varepsilon}(y)=\langle p_{\varepsilon}(y-\cdot),\phi\rangle_{\mathcal{P}(\mathbb{R}^d)}.$$

Let $\beta=(\beta_t)_{t\geq 0}$ be a fBm of index H. Then

$$\begin{aligned} J_{\delta}(s) &= =\frac{1}{\delta}\alpha_H\int_{s-\delta}^{s}\int_0^a|r-r'|^{2H-2}drdr'=\frac{1}{\delta}E[(\beta_s-\beta_{s-\delta})\beta_a]\\ &= \frac{1}{2\delta}[s^{2H}+|s-\delta-a|^{2H}-|s-a|^{2H}-|s-\delta|^{2H}]. \end{aligned}$$

Using an argument similar to (5.13) of [18], one can show that:

$$J_{\delta}(s)\leq 2Hs^{2H-1}+c_{a,H},\quad\forall s\in[0,t],\forall\delta>0, \tag{54}$$

where $c_{a,H}=H(a^{2H-1}+1)$. (This argument is based on treating separately the cases $a>s, a\leq s$, and considering in each case several intervals for δ.)

We claim that:

$$|I_{\varepsilon}(y)|\leq c_{\phi},\quad\forall y\in\mathbb{R}^d,\forall\varepsilon>0, \tag{55}$$

where c_{ϕ} is a constant depending on d, f and ϕ. To see this, we assume without loss of generality that $\phi=\psi * p_b$, for some $\psi\in C_0^{\infty}(\mathbb{R}^d), b>0$ (since functions of this form are dense in $\mathcal{P}(\mathbb{R}^d)$). Then, assuming that $\mu(d\xi)=g(\xi)d\xi$, we have:

$$\begin{aligned} |I_{\varepsilon}(y)| &= \left|\int_{\mathbb{R}^d}\overline{\mathcal{F}p_{\varepsilon}(y-\cdot)(\xi)}\mathcal{F}\phi(\xi)g(\xi)d\xi\right|\leq\int_{\mathbb{R}^d}|\mathcal{F}p_{\varepsilon}(y-\cdot)(\xi)|\ |\mathcal{F}\phi(\xi)|g(\xi)d\xi\\ &= (2\pi)^{-2d}\int_{\mathbb{R}^d}e^{-\varepsilon|\xi|^2/2}|\mathcal{F}\psi(\xi)|e^{-b|\xi|^2/2}g(\xi)d\xi\\ &\leq (2\pi)^{-2d}\int_{\mathbb{R}^d}|\mathcal{F}\psi(\xi)|e^{-b|\xi|^2/2}g(\xi)d\xi\\ &\leq (2\pi)^{-2d}\left(\int_{\mathbb{R}^d}|\mathcal{F}\psi(\xi)|^2g(\xi)d\xi\right)^{1/2}\left(\int_{\mathbb{R}^d}e^{-b|\xi|^2}g(\xi)d\xi\right)^{1/2}\\ &= (2\pi)^{-3d/2}\|\psi\|_{\mathcal{P}(\mathbb{R}^d)}\ [\psi^{(1)}(b,b)]^{1/2}:=c_{\phi}, \end{aligned}$$

where we used the Cauchy-Schwartz inequality for the last inequality above.

Combining (54) and (55), we get:

$$\sup_{(s,y)\in[0,t]\times\mathbb{R}^d}\ \sup_{\varepsilon,\delta>0}|V^{\varepsilon,\delta}(s,y)|\leq c_\phi(2Hs^{2H-1}+c_{a,H}). \tag{56}$$

From (53) and (56), we infer that for any $\varepsilon>0,\delta>0,(s,y)\in[0,t]\times\mathbb{R}^d$,

$$|p_{t-s}(x-y)E(u^{\varepsilon,\delta}_{s,y}F_\varphi)V^{\varepsilon,\delta}(s,y)|\leq\Psi(s,y),$$

where $\Psi(s,y):=K^*_t c_\phi p_{t-s}(x-y)(2Hs^{2H-1}+c_{a,H})$. Finally, we note that Ψ is integrable on $[0,t]\times\mathbb{R}^d$, since:

$$\begin{aligned}\int_0^t\int_{\mathbb{R}^d}\Psi(s,y)dyds &= K^*_t c_\phi\int_0^t(2Hs^{2H-1}+c_{a,H})\left(\int_{\mathbb{R}^d}p_{t-s}(x-y)dy\right)ds\\ &= K^*_t c_\phi(t^{2H}+c_{a,H}t)<\infty.\end{aligned}$$

□

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A Correction to Theorem 3.13 of [1]

Theorem 3.13 of [1] gives the necessary and sufficient condition for the existence of the solution of the equation: $u_t=\frac{1}{2}\Delta u+\dot{W}$ in $(0,T)\times\mathbb{R}^d$, with $u(0,\cdot)=0$. This condition is equivalent to saying that $\|g_{tx}\|_{\mathcal{HP}}<\infty$, where $g_{tx}(s,y)=[2\pi(t-s)]^{-d/2}\exp\{-|x-y|^2/[2(t-s)]\}=p_{t-s}(x-y)$. The condition is incorrectly stated in the case of the Bessel kernel, the heat kernel, and the Poisson kernel. We state below the correction of this result, whose proof will appear as an erratum in [2]. In connection with the present article, we observe that:

$$\|g_{tx}\|^2_{\mathcal{HP}}=\alpha_H\int_0^t\int_0^t|r-s|^{2H-2}I_f(r,s)drds=\alpha_1(t),$$

since $I_f(r,s):=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}g_{tx}(s,y)f(y-z)g_{tx}(r,z)dydz=\psi^{*(1)}(r,s)$.

Theorem A.1 *(i) If f is the Riesz kernel of order α, or the Bessel kernel of order α, then $\|g_{tx}\|_{\mathcal{HP}}<\infty$ if and only if $H>(d-\alpha)/4$.*

(ii) If f is the heat kernel of order α, or the Poisson kernel of order α, then $\|g_{tx}\|_{\mathcal{HP}}<\infty$ for any $H>1/2$ and $d\geq 1$.

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